Experimental investigation of dynamical invariants in bipartite entanglement

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The nonconservation of entanglement, when two or more particles interact, sets it apart from other dynamical quantities, like energy and momentum, and does not allow the interpretation of the subtle dynamics of entanglement as a flow of this quantity between the constituents of the system. Nevertheless, we show that when the interaction between a qubit and its environment is described by an amplitude-decay channel, the conservation of the mean excitation number and the inclusion of a third party leads to an invariant expression involving the bipartite entanglement between each of the parties and the other two. We provide an experimental demonstration of this idea using entangled photons and generalize it to N-partite Greenberger-Horne-Zeilinger states.

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I. INTRODUCTION

Proper understanding of the production, quantification, and evolution of quantum entanglement has been a major challenge of quantum physics, with direct implications on the relevance of this resource for applications in quantum information. An emblematic problem concerns the decay of initially entangled states under the influence of independent reservoirs acting on each part of the system. While each of these parts undergoes a typical decoherence process, affecting the populations and the coherences of the state, the dynamics of entanglement may differ considerably from local dynamics [1–13].

Usually, entanglement is not a conserved quantity. For instance, when an initially excited atom decays, releasing a photon into a zero-temperature environment, the initial and final states of the atom-environment system are not entangled, but the atom does get entangled to the environment at intermediate times. This process can be properly described by studying a qubit under the action of an amplitude-decay channel. We show, nevertheless, that, in this case, adding a third party to this two-party system (qubit + environment) leads to a dynamical invariance involving quantities that measure the bipartite entanglement between each part of the system and the other two parts. We show that this invariant is related to conservation of the number of excitations and to the Coffman-Kundu-Wootters (CKW) monogamy relation [14]. Entanglement monogamy and complementarity were studied experimentally in the quantum simulation of a frustrated Heisenberg spin system, with four photons [15]. Entanglement monogamy leads to complementarity relations between quantities related to parts of the system (like energy and visibility) and the amount of entanglement in the state [16–20]. Here we show yet another consequence of entanglement monogamy, an invariance relation valid for the paradigmatic amplitude-decay channel, and generalizable to N parties.

We demonstrate this entanglement invariance experimentally using the polarization of twin photons to implement the qubits and an optical interferometer to implement the amplitude-decay channel for one of them. This experimental configuration was used to demonstrate a dynamical law for the entanglement of a two-qubit system [12]. However, in that case the degrees of freedom of the environment were traced out, and the invariance of entanglement was not an issue. In the present case we perform measurements on the environment in order to verify the invariance of the bipartite entanglement between three parties: two qubits in the polarizations of two photons, and one in the environment. We also point out a generalization of this conservation law to N-partite Greenberger-Horne-Zeilinger (GHZ) states.

In Sec. II, we derive a conservation relation for the purity of each of the two subsystems, the system and the reservoir. In Sec. III, we show that, by adding a third system, the conservation of purity leads to a dynamical invariant relating the bipartite entanglement between each part and the other two. Section IV discusses the relation between these results and the CKW expression. Experimental results are exhibited in Sec. V, while our conclusions are summarized in Sec. VI.

II. INVARIANT IN TWO-QUBIT DYNAMICS INDUCED BY AN AMPLITUDE-DAMPING CHANNEL

Let S be a qubit and R the reservoir with which it interacts after $t = 0$. The initial product state of the S-R system is assumed to be $\rho_{SR}(0) = \rho_S(0) \otimes \rho_R(0)$, with

$$\rho_S(0) = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}, \quad \rho_R(0) = |\phi_0\rangle\langle\phi_0|. \tag{1}$$

The matrix ρ_S is written in the basis $\{|g\rangle, |e\rangle\}$ of the ground and excited states of S, and $|\phi_0\rangle$ stands for the ground state of the reservoir R. At $t = 0$ both S and R start to interact in such a way that the following transformation holds:

$$\begin{aligned} |g\rangle|\phi_0\rangle &\rightarrow |g\rangle|\phi_0\rangle, \\ |e\rangle|\phi_0\rangle &\rightarrow \sqrt{1-p}|e\rangle|\phi_0\rangle + \sqrt{p}|g\rangle|\phi_1\rangle, \end{aligned} \tag{2}$$

where $p = p(t) \in [0,1]$ is a time-dependent parameter, and $|\phi_1\rangle$ denotes the first excited state (orthogonal to $|\phi_0\rangle$) of R. The map (2) corresponds precisely to the amplitude-damping channel. For different parametrizations $p(t)$, the transformation (2) represents several physical processes such as the spontaneous emission of a photon by a two-level atom in a zero-temperature electromagnetic environment [where $p(t=0)=0$, $p(t\to\infty)=1$], or the interaction of a two-level atom with a single mode of the electromagnetic field inside a cavity. According to map (2), matrices (1) evolve into

$$\rho_S(p) = \begin{pmatrix} 1-\rho_{ee}(1-p) & \rho_{ge}\sqrt{1-p} \\ \rho_{eg}\sqrt{1-p} & \rho_{ee}(1-p) \end{pmatrix},$$
$$\rho_R(p) = \begin{pmatrix} 1-\rho_{ee}p & \rho_{ge}\sqrt{p} & \cdots \\ \rho_{eg}\sqrt{p} & \rho_{ee}p & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}, \tag{3}$$

where "$\cdots$" in the expression for $\rho_R(p)$ represents empty rows and columns corresponding to the infinite remaining null matrix elements. Inspection of the reduced density matrices (3) shows that the information initially contained in the system S is transferred during the evolution to the system R. The transfer is complete at $p=1$, when the states of S and R become exchanged.

An important observation regarding the dynamics imposed by transformation (2) is that the mean number $\langle\hat{N}\rangle$ of total excitations,

$$\langle\hat{N}\rangle = \langle\hat{n}_S(p) + \hat{n}_R(p)\rangle, \tag{4}$$

is conserved throughout the entire evolution, thus restricting the way the populations (in the referred basis) are transferred. Here $\hat{n}_S$ and $\hat{n}_R$ are the excitation-number operators of systems S and R, respectively. For the specific case in which S represents a two-level atom interacting with one mode of the electromagnetic field in a cavity, these operators are given by $\hat{n}_S = \frac{1}{2}(\mathbb{I} - \sigma_z)$ and $\hat{n}_R = \hat{a}^\dagger\hat{a}$.

Quite generally, the conservation of $\langle\hat{N}\rangle$ follows immediately from the expressions for $\langle\hat{n}_R(p)\rangle$ and $\langle\hat{n}_S(p)\rangle$,

$$\langle\hat{n}_S(p)\rangle = \mathrm{Tr}[\rho_S(p)\hat{n}_S] = \rho_{ee}(1-p),$$
$$\langle\hat{n}_R(p)\rangle = \mathrm{Tr}[\rho_R(p)\hat{n}_R] = \rho_{ee}p, \tag{5}$$

so that $\langle\hat{N}\rangle = \rho_{ee}$.

In the following, we investigate how conservation (4) manifests itself in the evolution of the purity $\pi_i \equiv \mathrm{Tr}\rho_i^2$ of the subsystem $i = S,R$. With the aid of Eqs. (3) and (5) it is straightforward to show that $\pi_i(p)$ can be written as

$$\pi_i(p) = 2\langle\hat{n}_i(p)\rangle^2 - 2\langle\hat{n}_i(p)\rangle\Lambda + 1, \tag{6}$$

where $\Lambda = 1 - |\rho_{ge}|^2/\rho_{ee}$. Inverting this expression, we obtain

$$\langle\hat{n}_i(p)\rangle = \frac{\Lambda}{2} \pm \frac{1}{2}\sqrt{\Lambda^2 - 2[1-\pi_i(p)]}. \tag{7}$$

Hence, we can rewrite conservation equation (4) as

$$\rho_{ee} = \Lambda \pm \mathcal{W}_S(\Lambda,p) \pm \mathcal{W}_R(\Lambda,p), \tag{8}$$

where we have defined $\mathcal{W}_i(\Lambda,p) = \frac{1}{2}\sqrt{\Lambda^2 - 2[1-\pi_i(p)]}$. As seen in Eq. (7), the appropriate choice of the sign in front of $\mathcal{W}_i$ depends on whether $\langle\hat{n}_i(p)\rangle$ is greater or smaller than $\Lambda/2$. If the initial value ρ_{ee} is smaller than or equal to $\Lambda/2$, then the restriction $\langle\hat{n}_i(p)\rangle \leqslant \rho_{ee}$ implies that $\langle\hat{n}_i(p)\rangle \leqslant \Lambda/2$, and both minus signs should be taken in Eq. (8). On the other hand, if ρ_{ee} is larger than $\Lambda/2$, then the signs in Eq. (8) depend on the value of p. From Eqs. (5) it follows that $\langle\hat{n}_S(p)\rangle \gtrless \Lambda/2$ whenever $p \lessgtr 1 - \Lambda/2\rho_{ee}$, whereas $\langle\hat{n}_R(p)\rangle \gtrless \Lambda/2$ for every $p \gtrless \Lambda/2\rho_{ee}$. In both cases the upper or lower inequality sign determines the $\pm$ sign that should be used.

III. THE RELEVANT ROLE OF A THIRD PARTY: ENTANGLEMENT INVARIANT

With the previous results we see that, as a consequence of the conservation of $\langle\hat{N}\rangle$, the purities $\pi_S(p)$ and $\pi_R(p)$ evolve in such a way that the right-hand side of Eq. (8) remains constant throughout the evolution. It is worth noticing that the factor $2[1-\pi_i(p)]$, which naturally arises in Eq. (7), is precisely the square of the concurrence whenever the system i results from tracing a pure bipartite state [21]. In this case, $2(1-\pi_i)$ quantifies the bipartite entanglement between the qubit i and the rest of the system. Therefore, it is convenient to include a third qubit (M) in order to purify the global state. The introduction of M then allows us to interpret the term $2(1-\pi_i)$ as a measure of bipartite entanglement [alternatively, we may use a measure based on the Schmidt weight K_i [22], which is related to π_i according to $\pi_i(p) = K_i^{-1}(p)$], and hence to use Eq. (7) to throw some light on the global dynamical properties of (bipartite) entanglement for systems undergoing the interaction modeled by map (2). Let us then suppose that $\rho_S(0)$ results from partial tracing over the system M on the pure general state

$$|\psi(0)\rangle = \alpha|M_1\rangle|e\rangle + \beta|M_0\rangle|g\rangle + \gamma|M_1\rangle|g\rangle + \delta|M_0\rangle|e\rangle, \tag{9}$$

where $|M_0\rangle$, $|M_1\rangle$ are two orthogonal states of M. In this case the elements of the initial density matrix $\rho_S(0)$ are $\rho_{ee} = |\alpha|^2 + |\delta|^2$, $\rho_{ge} = \beta\delta^* + \alpha^*\gamma$, and $\rho_{gg} = |\beta|^2 + |\gamma|^2$. At $t=0$ we allow system S to interact with the environment, according to transformation (2). Then, the initial tripartite state $|\Psi(0)\rangle = |\psi(0)\rangle|\phi_0\rangle$ evolves to

$$\begin{aligned}|\Psi(p)\rangle &= \alpha\,|M_1\rangle(\sqrt{1-p}\,|e\rangle|\phi_0\rangle + \sqrt{p}\,|g\rangle|\phi_1\rangle) \\ &\quad + \delta\,|M_0\rangle(\sqrt{1-p}\,|e\rangle|\phi_0\rangle + \sqrt{p}\,|g\rangle|\phi_1\rangle) \\ &\quad + (\beta\,|M_0\rangle + \gamma\,|M_1\rangle)\,|g\rangle|\phi_0\rangle\,. \end{aligned}\tag{10}$$

Using Eq. (10), the density matrix $\rho_{\mathrm{MSR}}(p)$ of the complete system may be constructed, and the reduced density matrices $\rho_i(p)$ with $i = M,R,S$ can be computed. Since M does not interact at all, ρ_M is constant and given by

$$\rho_M = \begin{pmatrix} |\beta|^2 + |\delta|^2 & \beta\gamma^* + \alpha^*\delta \\ \alpha\delta^* + \beta^*\gamma & |\alpha|^2 + |\gamma|^2 \end{pmatrix}. \tag{11}$$

We return to Eq. (8) and observe that, given that it was obtained from a property of the interaction between R and S

only, it remains valid even when the system M is considered. Moreover, as seen from Eq. (10), the coefficients (α and δ) that determine ρ_{ee} are precisely the coefficients responsible for the entanglement between M and the rest of the system. This is an important observation since it relates ρ_{ee} directly to such entanglement and hence allows us to relate ρ_{ee} with the (constant) purity of the system M as follows:

$$\begin{aligned}\rho_{ee} &= \frac{\Lambda}{2} \pm \frac{1}{2}\sqrt{\Lambda^2 - 2(1-\pi_M)} \\ &= \frac{\Lambda}{2} \pm \mathcal{W}_M(\Lambda).\end{aligned} \quad (12)$$

Once more, the $\pm$ sign depends on whether ρ_{ee} is larger (plus sign) or smaller (minus sign) than $\Lambda/2$. Introducing this last expression into Eq. (8) leads to

$$\pm\mathcal{W}_M(\Lambda) - \frac{\Lambda}{2} = \pm\mathcal{W}_S(\Lambda,p) \pm \mathcal{W}_R(\Lambda,p). \quad (13)$$

According to the discussion at the beginning of this section, since the tripartite state $|\Psi(0)\rangle$ is pure we can write $2[1-\pi_i(p)] = C^2_{i(jk)}$, where $C^2_{i(jk)}$, known as the *tangle*, stands for the square of the concurrence $C_{i(jk)}$, which measures the bipartite entanglement between the system i and the other two subsystems (jk) considered as a whole. Hence, $\mathcal{W}_i$ becomes $\mathcal{W}_i(\Lambda,p) = \frac{1}{2}\sqrt{\Lambda^2 - C^2_{i(jk)}(p)}$ and Eq. (13) takes the form

$$\begin{aligned}&\pm\sqrt{\Lambda^2 - C^2_{M(SR)}} - \Lambda \\ &\quad = \pm\sqrt{\Lambda^2 - C^2_{S(MR)}(p)} \pm \sqrt{\Lambda^2 - C^2_{R(MS)}(p)}.\end{aligned} \quad (14)$$

As stated above, the appropriate choice of signs for each $\mathcal{W}_i$ is determined by the magnitude of ρ_{ee} relative to $\Lambda/2$, as well as by the value of p. An inspection of all the valid combinations leads to the following cases (we omit the dependence on Λ):

$$\frac{\Lambda}{2} + \mathcal{W}_M = \mathcal{W}_S(p) + \mathcal{W}_R(p), \quad p \in [0,1], \quad (15)$$

whenever $\rho_{ee} \leqslant \frac{\Lambda}{2}$, and

$$\frac{\Lambda}{2} - \mathcal{W}_M = \begin{cases}\mathcal{W}_R(p) - \mathcal{W}_S(p), & p < 1 - \frac{\Lambda}{2\rho_{ee}}, \\ \mathcal{W}_S(p) + \mathcal{W}_R(p), & p \in \left[1 - \frac{\Lambda}{2\rho_{ee}}, \frac{\Lambda}{2\rho_{ee}}\right], \\ \mathcal{W}_S(p) - \mathcal{W}_R(p), & p > \frac{\Lambda}{2\rho_{ee}},\end{cases} \quad (16)$$

whenever $\rho_{ee} > \frac{\Lambda}{2}$. Equations (15) and (16) stand as conservation relations involving bipartite entanglement terms along the evolution described by map (2). We see that, once the original S-R system is enlarged to include system M needed for purification, conservation (4) acquires a new significance in terms of bipartite entanglement. Thus, while the initial entanglement between S and M turns into entanglement between M and R, the entanglement distributes in such a way that the invariant relations (15) and (16) hold.

In the particular case in which the initial density matrix $\rho_S(0)$ is diagonal so that $\Lambda = 1$ [for example, if $\delta = \gamma = 0$ in Eq. (9)], then the invariant relations (15) and (16) involve the quantities $W_i(p) \equiv \mathcal{W}_i(1,p)$, which are explicitly written as

$$\begin{aligned}W_S(p) &= \left|\rho_{ee}(1-p) - \tfrac{1}{2}\right|, \\ W_R(p) &= \left|\rho_{ee}p - \tfrac{1}{2}\right|, \\ W_M &= \left|\rho_{ee} - \tfrac{1}{2}\right|,\end{aligned} \quad (17)$$

as follows from the definition of W_i and Eqs. (5) and (6).

The conservation relations (15) and (16) for $\Lambda = 1$ hold even when we consider the system M to be in an initial state entangled with N qubits S_j ($j = 1,2,\ldots,N$). This allows us to find new global quantities that are conserved during the evolution. For example, the expression equivalent to Eq. (15) for the $N+1$ GHZ-type state

$$|\chi_N\rangle = \alpha|M_1\rangle\Pi_j^N|e_j\rangle + \beta|M_0\rangle\Pi_j^N|g_j\rangle \quad (18)$$

is

$$W_M + \frac{1}{2} = \frac{1}{N}\sum_{j=1}^{N}\left[W_{S_j}(p_j) + W_{R_j}(p_j)\right]. \quad (19)$$

IV. ENTANGLEMENT INVARIANT AND THE CKW RELATION

In the preceding sections we showed how the conservation of $\langle\hat{N}\rangle$ entails information regarding the dynamics of bipartite entanglement. Such information is provided by Eq. (14), which shows an invariant quantity that relates the bipartite entanglement (measured by $C^2_{i(jk)}$) of each of the qubits with the remaining two. The fact that the different tangles $C^2_{i(jk)}$ are not independent is well known and follows from a relation derived by Coffman, Kundu, and Wooters [14]:

$$C^2_{i(jk)} = C^2_{ij} + C^2_{ik} + \tau_{ijk}. \quad (20)$$

Here τ_{ijk} (or three-tangle) represents the genuine (multipartite) entanglement shared by the three systems i, j, and k. Equation (20) couples the tangles $C^2_{M(SR)}$, $C^2_{S(RM)}$, and $C^2_{R(MS)}$ and therefore establishes a relation between them. Hence, a question arises as to what extent Eqs. (20) and (14) are related. It is clear from the previous section that the CKW relation is not necessary for the invariant (14) to emerge [nowhere did we resort to Eq. (20) to obtain Eq. (14)]. Thus, in what follows we investigate the connection between Eqs. (20) and (14), considering the transformation given by Eq. (2). This serves not only to prove the consistency of our results, but also to point out some important differences between both approaches.

We start by determining the value of τ_{ijk} for the state $|\Psi(p)\rangle$ obtained by applying transformation (2) to the initial state $|\Psi(0)\rangle = |\psi(0)\rangle|\phi_0\rangle$. As shown in Ref. [14], if the three-qubit system is the pure state $|\varphi\rangle_{\rm MSR} = \sum_{nlm} a_{nlm}|nlm\rangle_{\rm MSR}$ (here $n,l,m = 0,1$), then τ_{ijk} is given by (a sum over repeated indices is assumed)

$$\tau_{ijk} = 2|a_{nlm}a_{n'l'r}a_{sqm'}a_{s'q'r'}\epsilon_{nn'}\epsilon_{ll'}\epsilon_{mm'}\epsilon_{rr'}\epsilon_{ss'}\epsilon_{qq'}|. \quad (21)$$

Direct substitution of the coefficients of Eq. (10) shows that for the state $|\Psi(p)\rangle$ the three-tangle vanishes, so the CKW relation reduces to

$$C^2_{i(jk)} = C^2_{ij} + C^2_{ik}, \quad (22)$$

which can be used to obtain

$$C^2_{M(SR)} = C^2_{S(MR)} + C^2_{R(MS)} - 2C^2_{SR}. \tag{23}$$

The last term in this equation reveals that a relation involving only the tangles $C^2_{M(SR)}$, $C^2_{S(RM)}$, and $C^2_{R(MS)}$ will not be linear. In the next step, we would like to rewrite C^2_{SR} as a function of $C^2_{S(RM)}$ and $C^2_{R(MS)}$. On the one hand, we notice that, with the aid of Eq. (6), $C^2_{S(RM)}$ and $C^2_{R(MS)}$ can be expressed as (here $i = S,R$)

$$C^2_{i(jk)} = 2(1-\pi_i) = 4\langle\hat{n}_i\rangle(\Lambda - \langle\hat{n}_i\rangle), \tag{24}$$

so that

$$\sqrt{\Lambda^2 - C^2_{S(MR)}} = |2\langle\hat{n}_S\rangle - \Lambda|,$$
$$\sqrt{\Lambda^2 - C^2_{R(MS)}} = |2\langle\hat{n}_R\rangle - \Lambda|. \tag{25}$$

On the other hand, a direct calculation of C^2_{SR}—the entanglement between S and R created directly as a result of the amplitude-damping channel—using Eqs. (5) leads to

$$C^2_{SR} = 4\langle\hat{n}_S\rangle\langle\hat{n}_R\rangle. \tag{26}$$

Equations (25) and (26) show that $\langle\hat{n}_S\rangle$ and $\langle\hat{n}_R\rangle$ are the variables that naturally relate C^2_{SR} with $C^2_{S(RM)}$ and $C^2_{R(MS)}$, and hence they constitute the natural quantities to be used to determine the final relation between $C^2_{M(SR)}$, $C^2_{S(RM)}$, and $C^2_{R(MS)}$ (this result is expected from the previous section). From Eqs. (25) we have that (recall that $\mathcal{W}_i = \frac{1}{2}\sqrt{\Lambda^2 - C^2_{S(MR)}}$)

$$\langle\hat{n}_S\rangle = \pm\mathcal{W}_S + \frac{\Lambda}{2},$$
$$\langle\hat{n}_R\rangle = \pm\mathcal{W}_R + \frac{\Lambda}{2}, \tag{27}$$

where the $\pm$ signs before each $\mathcal{W}_i$ depends on whether $\langle\hat{n}_i\rangle \gtrless \Lambda/2$. Substitution of Eqs. (27) into Eq. (26) leads us to rewrite Eq. (23) (rewritten in terms of $\mathcal{W}$'s) in the form

$$\mathcal{W}_M^2 = \mathcal{W}_S^2 + \mathcal{W}_R^2 + 2\left[\left(\pm\mathcal{W}_S + \frac{\Lambda}{2}\right)\left(\pm\mathcal{W}_R + \frac{\Lambda}{2}\right)\right] - \frac{\Lambda^2}{4}. \tag{28}$$

Rearranging terms we arrive at

$$\mathcal{W}_M^2 = \left(\pm\mathcal{W}_S \pm \mathcal{W}_R + \frac{\Lambda}{2}\right)^2 \tag{29}$$

or $\mathcal{W}_M = |\pm\mathcal{W}_S \pm \mathcal{W}_R + \frac{\Lambda}{2}|$ and hence

$$\pm\mathcal{W}_M - \frac{\Lambda}{2} = \pm\mathcal{W}_S \pm \mathcal{W}_R, \tag{30}$$

which is precisely Eq. (13).

Since Eq. (30) is just an alternative way of expressing Eq. (23)—which arises directly from Eq. (22)—we can conclude that the CKW relation (applied to the present case) indeed leads to the invariant (14). However, an important distinction between this approach and the former derivation of Eq. (13) must be pointed out. It refers to the fact that the invariance in Eq. (30) is interpreted as due to the conservation of $\mathcal{W}_M$ through the evolution, while $\mathcal{W}_S$ and $\mathcal{W}_R$ evolve with p [recall that ρ_M and, hence, $C^2_{M(SR)}$ remain constant during the evolution, in contrast to $C^2_{S(MR)}(p)$ and $C^2_{R(MS)}(p)$]. Thus, if we make use of the CKW relation *alone* to derive Eq. (30), the *physical* origin of the invariance remains hidden. However, the derivation performed in the previous section shows, in a very transparent way, that the invariance is a consequence of the conservation of the mean number of total excitations, $\langle\hat{N}\rangle$. In fact, we can verify that a constant value of $\mathcal{W}_M$ implies the conservation of $\langle\hat{N}\rangle$. To demonstrate that, it is enough to use Eqs. (24) and (26) to rewrite Eq. (23) as

$$\begin{aligned} C^2_{M(SR)} &= 4\langle\hat{n}_S\rangle(\Lambda - \langle\hat{n}_S\rangle) \\ &\quad + 4\langle\hat{n}_R\rangle(\Lambda - \langle\hat{n}_R\rangle) - 8\langle\hat{n}_S\rangle\langle\hat{n}_R\rangle \\ &= -[2(\langle\hat{n}_S\rangle + \langle\hat{n}_R\rangle) - \Lambda]^2 + \Lambda^2, \end{aligned} \tag{31}$$

so that an equation analogous to Eq. (25) arises:

$$\Lambda^2 - C^2_{M(SR)} = 2\mathcal{W}_M = (2\langle\hat{N}\rangle - \Lambda)^2. \tag{32}$$

This form makes it clear that the conservation of $\mathcal{W}_M$ (or equivalently of $C^2_{M(SR)}$) has its physical root in the conservation of $\langle\hat{N}\rangle$. Moreover, with Eq. (32) at hand, we could resort to Eq. (22) and claim that

$$C^2_{M(SR)} = C^2_{MS}(p) + C^2_{MR}(p), \tag{33}$$

which is an invariant quantity much simpler than Eq. (14). However, we note that this invariant is expressed in terms of qubit-qubit concurrences, which always involve system M. It does not include the term C^2_{SR}, which bears information regarding the dynamics of the entanglement, that is locally created as a result of transformation (2). In contrast, when substituting $C^2_{i(jk)} = C^2_{ij} + C^2_{ik}$ in Eq. (14), it stands as an invariant in which all the three contributions C^2_{MS}, C^2_{MR}, and C^2_{SR} intervene explicitly.

A further aspect that distinguishes these two approaches is that the CKW result stems from the structure of the Hilbert space for the three-qubit system, while our results appear as a consequence of the conservation of the excitation number in a dynamical evolution.

V. EXPERIMENTAL RESULTS

We verified the validity of Eqs. (15) and (16) experimentally, using polarization-entangled photons generated from spontaneous parametric down-conversion (SPDC). The polarization entanglement is prepared with a two-crystal source [23]. The experimental setup is shown in Fig. 1. A 325-nm cw He-Cd laser is used to pump two 1-mm-long type-I β-barium borate (BBO) crystals. The down-converted photons are spectrally filtered with 10-nm-bandwidth interference filters, and spatially filtered through 2-mm detection apertures, before detection with single-photon counting modules. Identifying the polarization state of photon 2 as the system S and the polarization state of photon 1 as the third party M, the source is set up to produce the initial state

$$|\Phi\rangle = \alpha|V\rangle_M|V\rangle_S + \beta|H\rangle_M|H\rangle_S.$$

Let us further identify the longitudinal spatial mode of photon 2 as the reservoir R, and call the initial spatial mode $|0\rangle$. A displaced Sagnac interferometer with a nested wave plate

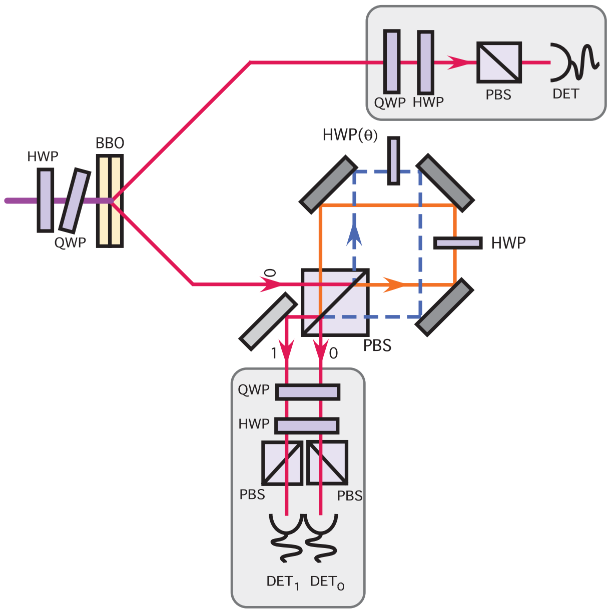

FIG. 1. (Color online) Experimental setup. An entangled state is prepared in the polarization of a photon pair. One of the photons is sent through a Sagnac-like interferometer that implements an amplitude-decay channel. Quantum-state tomography is used to reconstruct the polarization state of the photon pair, detected in coincidence.

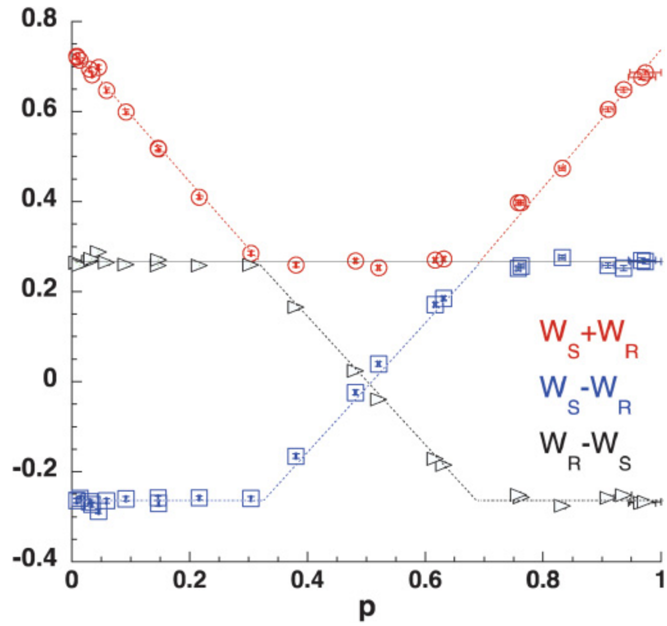

FIG. 2. (Color online) Experimental results for $\rho_{ee} = 0.73$, $\Lambda = 1$. The lines correspond to the functions $W_R(p) - W_S(p)$ (triangles), $W_S(p) + W_R(p)$ (circles), and $W_S(p) - W_R(p)$ (squares). The continuous line represents the invariant I_{SR}.

can be used to implement the following transformation on the polarization and spatial mode of photon 1 [10,12]:

$$|H\rangle|0\rangle \rightarrow |H\rangle|0\rangle,$$
$$|V\rangle|0\rangle \rightarrow \cos\theta|V\rangle|0\rangle + \sin\theta|H\rangle|1\rangle, \tag{34}$$

where $|0\rangle$ and $|1\rangle$ refer to different spatial modes and θ is twice the angle of the half-wave plate. It has been demonstrated with quantum process tomography that this interferometer implements the amplitude-damping channel with fidelities as high as $\sim$0.95 [12]. Identifying the polarization states $\{|H\rangle,|V\rangle\}$ with the system states $\{|g\rangle,|e\rangle\}$, the spatial modes $\{|0\rangle,|1\rangle\}$ with the reservoir states $\{|\phi_0\rangle,|\phi_1\rangle\}$, and $p = \sin^2\theta$, transformation (34) becomes equivalent to transformation (2), provided $0 \leqslant \theta \leqslant \pi/2$. The initial state $|\Psi\rangle_{SM}|0\rangle_R$ evolves to

$$|\Phi(\theta)\rangle = (\beta|H\rangle_M|H\rangle_S + \alpha\cos\theta|V\rangle_M|V\rangle_S)|0\rangle_R + \alpha\sin\theta|V\rangle_M|H\rangle_S|1\rangle_R, \tag{35}$$

after propagation through the interferometer. The final state is equivalent to $|\Psi(p)\rangle$ given in Eq. (10) with $\gamma = \delta = 0$. It is important to note that partially tracing over any two of the three subsystems of state (35) leads to a diagonal reduced density matrix. Then, the purities π_j (or equivalently, the Schmidt weights K_j) and each W_j $(j = S,M,R)$ term in Eq. (15) can be determined by local population measurements, made with the detectors 0 and 1 shown in Fig. 1.

By rotating the half-wave plate (HWP) in the pump beam, we selected different values of α and β. For instance, we selected $|\alpha|^2 = \rho_{ee} = 0.31$, 0.5, and 0.73 with corresponding purities 0.97, 0.94, and 0.96, so that the invariant relations could be applied directly. These purities were calculated from quantum-state tomography of the initial states (after passage through the interferometer with $\theta = 0$; see below). Imperfect purity is probably due to spatial walkoff in the crystal and imperfect alignment of the interferometer. These parameters indicate that the experimental state is quite close the the ideal initial pure state.

Projective measurements on S and M are performed using wave plates and polarizing beam splitters to project onto polarization states, while projection onto spatial modes is performed by placing detectors in mode 0 or 1. We performed measurements for several values of $p = \sin^2\theta$ $(0 \leqslant \theta \leqslant \pi/2)$ characterizing the amplitude-damping channel in transformation (34).

Figure 2 shows the theoretical curves and the experimental data of each of the three functions that define the invariant $I_{SR} = \frac{\Lambda}{2} - \mathcal{W}_M$, according to Eq. (16) for the case $\rho_{ee} = 0.73$ and $\Lambda = 1$. The triangles correspond to $W_R(p) - W_S(p)$, the circles to $W_S(p) + W_R(p)$, and the squares to $W_S(p) - W_R(p)$. The invariant (piecewise) function I_{SR} is here represented by the continuous line and corresponds, as follows from Eq. (16) (with $\Lambda = 1$), to one of the above curves depending on whether p is in the interval $[0,1 - \frac{1}{2\rho_{ee}})$, $[1 - \frac{1}{2\rho_{ee}}, \frac{1}{2\rho_{ee}}]$, or $(\frac{1}{2\rho_{ee}},1]$.

Figure 3 shows both the theoretical and the experimental curves of the invariant quantity I_{SR} for three different values of ρ_{ee} (in all cases $\Lambda = 1$). The top (red) and middle (blue) curves, corresponding respectively to $\rho_{ee} = 0.31$ and $\rho_{ee} = 0.5$, represent the invariant sum $I_{SR} = \frac{1}{2} + W_M = W_S(p) + W_R(p)$ [see Eq. (15)], whereas the bottom (black) curve, corresponding to $\rho_{ee} = 0.73$, represents the invariant $I_{SR} = \frac{1}{2} - W_M$ given by expression (16). From Figs. 2 and 3 we see that the experimental data fit the theoretical curves within the precision of the measurements, thus demonstrating experimentally the conservation relations (15) and (16).

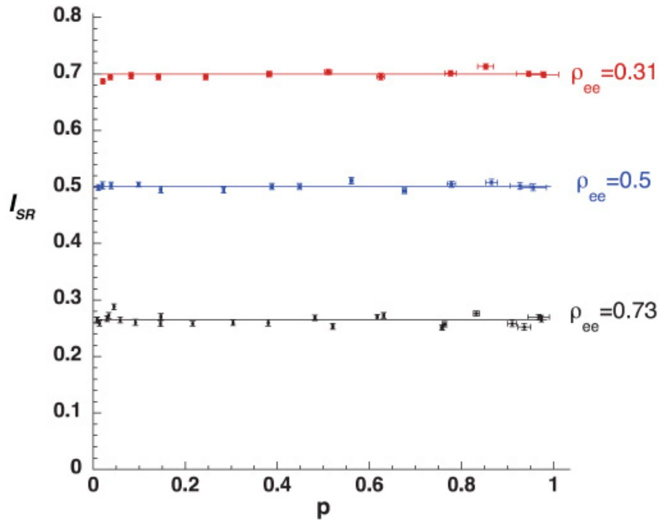

FIG. 3. (Color online) The invariant I_{SR} is plotted for different values of ρ_{ee}.

VI. CONCLUSIONS

We considered a two-qubit system in which one of them is subject to the action of an amplitude-damping channel, and the other is a two-level reservoir. We showed that the existence of a conserved quantity (characteristic of the transformation), together with the inclusion of a third party, leads to an invariant relation involving quantities that measure the bipartite entanglement between each qubit and the rest of the system. This relation was verified experimentally, using an optical setup based on entangled photons, which has the special advantage of allowing the precise monitoring of the environment.

The invariants discussed in this paper include the environment degrees of freedom, which are very hard to access in the majority of the physical implementations of two-level systems. Our setup, on the other hand, allows fine control and monitoring of all qubits involved, including the environment. Previous experiments realized with similar setups [10] have ignored the environmental degrees of freedom, which were traced out in order to obtain the reduced density matrix of the decaying system. Here the environment itself is monitored, leading to our entanglement invariance relation.

We generalized the invariant expression to N-partite GHZ states and discussed the relation between our result and the expression for the monogamy of entanglement derived by Coffman, Kundu, and Wooters. Our work shows in fact that the CKW relation leads to entanglement invariants when a two-level system decays under the action of an environment. This could be a first step in the search for more general dynamical invariants that restrict the evolution and dynamics of entanglement in multiparticle systems.

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