Robustness of entanglement as a resource

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The robustness of multipartite entanglement of systems undergoing decoherence is of central importance to the area of quantum information. Its characterization depends, however, on the measure used to quantify entanglement and on how one partitions the system. Here we show that the unambiguous assessment of the robustness of multipartite entanglement is obtained by considering the loss of functionality in terms of two communication tasks, namely the splitting of information between many parties and the teleportation of states.

I. INTRODUCTION

The fast development of quantum information in the last two decades has strengthened the notion that entanglement is not only a fundamental concept in quantum mechanics, but also a resource for processing and transmitting information. For bipartite systems, the relationship between entanglement measures and the use of entanglement for specific tasks like teleportation [1] and quantum key distribution [2] is well understood. However, the exact role of multipartite entanglement in communication protocols and in quantum computation is still an open question. In the multipartite context, many different measures and kinds of entanglement are possible [3]. But it is still unclear what is the relevance of these measures for quantifying the ability of performing different tasks, and what is the role of each kind of entanglement in surpassing the classical gain for a given protocol.

We show in this paper that these questions are particularly relevant for open systems. Indeed, different kinds of entanglement can behave quite differently under decoherence. Once the resource for a given protocol is (non)robust against perturbations, this implies that the protocol itself should also be (non)robust. In fact, we show here that the comparison between the effect of the environment on the many kinds of entanglement for a multipartite system, used as a resource for a given task, and the effect of decoherence on how well the task is achieved indicates which type of entanglement is relevant for accomplishing it.

The dynamics of entanglement, for a wide class of entangled states, have been extensively analyzed [4–11] and experimentally demonstrated [12,13]. The nonunitary evolution depends intrinsically on the system-environment dynamics and the quantum entangled state under consideration. Besides a more general treatment restricted to a two-qubit case given in [11,12], all analyses have restricted themselves to particular initial entangled states evolving under some specific dynamics [4–10]. These studies have shown that the dynamics of entanglement is quite distinct from the dynamics of populations and coherences. In particular, entanglement may vanish at a finite time, much before the asymptotic decay of the coherences [4–7].

For multipartite systems the problem gets more involved since there are inequivalent classes of entangled states [14], that is, states that are not connected by local operations and classical communication (LOCC's). States that can be obtained from each other through LOCC's belong to the same class of states, meaning that these states are equivalent resources for a large class of tasks in quantum information, those that are invariant under LOCC's, like teleportation [1] and distillation [15].

Many of the investigations on the open-system dynamics of entanglement refer to GHZ states [5–9]

$$|\text{GHZ}_N\rangle = \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}, \tag{1}$$

subject to the normalization condition $|\alpha|^2 + |\beta|^2 = 1$.

Another class of entangled states, the *W* states

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|00\cdots01\rangle + |01\cdots00\rangle + \cdots + |00\cdots01\rangle), \tag{2}$$

were introduced in [14], where it was shown that they are inequivalent to the GHZ states.

It is easy to see that the entanglement of *W* states is more robust than that of GHZ states upon loss of a particle [14]. Given this symptomatic entanglement robustness of *W* states, it is natural to ask which class of states, GHZ or *W*, is more robust in a more general situation where each part of a multipartite state is undergoing decoherence. The entanglement dynamics analysis of *W* and GHZ states is relevant on its own since these states are resources for quantum information protocols as quantum teleportation, dense coding, and secure distribution of quantum keys [16,17], and have been experimentally produced in many kinds of physical systems including atoms in cavities, trapped ions, and entangled photons [18,19].

The simple form of *N*-partite GHZ states has allowed several analytical derivations of the associated entanglement dynamics [5–7]. An interesting conclusion was drawn for multipartite GHZ states subjected to local environments [7]: The time at which bipartite (and multipartite) entanglement becomes arbitrarily small can occur well before the disentanglement time; this time difference strongly increases with the number of particles. This implies that, for GHZ states, the time for which entanglement becomes arbitrarily small better characterizes the entanglement robustness than the disentanglement time. On the other hand, the global entanglement of *W* states was shown to be more robust: The decay rate is size independent for dephasing and zero temperature environments [8,9]. This previous work motivates the question on how other types of entanglement, defined between different kinds of partitions of the state, behave

for W states under decoherence. The physical meaning of these different types of entanglement and the corresponding measures is also relevant, the important question being how do they relate to possible applications in computation and communication.

In this paper we analyze the original W state (2) and related states, referred to as W-like states

$$|\widetilde{W}_N\rangle = a_1|10\cdots00\rangle + a_2|01\cdots00\rangle + \cdots + a_N|00\cdots01\rangle,$$
(3)

$$\left|W_N^0\right\rangle = \alpha|W_N\rangle + \beta|0\rangle^{\otimes N},$$
(4)

$$\left|\widetilde{W}_N^0\right\rangle = \alpha|\widetilde{W}_N\rangle + \beta|0\rangle^{\otimes N}.$$
(5)

We show that, when considering bipartite entanglement, these states display the same kind of nonrobustness as the GHZ states. For two kinds of decoherence, we derive an analytical expression, valid for any entanglement measure, which can be expressed as the entanglement of that of considerably smaller subsystems, in the same spirit as in [10]. In the dephasing case, the bipartite and global entanglements in W states remain both very robust when the number of particles increases, the disentanglement time being equivalent in this case to the time at which entanglement becomes very small. Under the amplitude damping channel, these two time scales become quite distinct: The bipartite entanglement as quantified by the negativity [20], for a sufficiently large N, decays below an arbitrarily small value much before it vanishes, in spite of the fact that the global entanglement is size independent. This clearly shows how differently can distinct kinds of entanglement behave under decoherence. Adopting a more practical point of view and explicitly considering as tasks the teleportation [1] and the splitting [21] of quantum information, we show that the dynamics of fidelity for each of these protocols, when using as a resource an entangled state undergoing decoherence, can be identified with the decay of a specific kind of entanglement measure, thus allowing the identification of each of these measures with a well-defined task.

Our paper is organized as follows. In Sec. II, we introduce the decoherence maps to be considered in this article. In Sec. III, we discuss the entanglement quantifiers that are used to characterize the entanglement of multipartite states. In Sec. IV, we introduce a method that allows for the calculation of the entanglement corresponding to a given bipartition of a W or W-like multipartite state through the calculation of the entanglement of a two-qubit system, which greatly simplifies the analysis of the dynamics of entanglement for this kind of state. This technique is used in Sec. V to compare the dynamics of global and bipartite quantifiers of entanglement, which have been proposed in the literature. The quite different features under decoherence of these quantifiers is shown in Sec. VI to have a direct relation to the robustness of inequivalent communication tasks. In Sec. VII we summarize our conclusions. Detailed calculations are given in the Appendix.

II. DECOHERENCE MODELS

The dynamics of a system interacting with an environment can be described in the Kraus form [22] so that the density operator of the evolved system is given by $\rho = \sum_j E_j \rho_0 E_j^\dagger$,

where E_j are positive-operator-valued measures (POVM's) satisfying $\sum_j E_j^\dagger E_j = 1$, and ρ_0 is the initial state of the system. We analyze here the entanglement dynamics of the states (2), (3), (4), and (5), evolving under the action of two paradigmatic noisy channels: dephasing and amplitude damping. The states are described in the computational basis $\{|0\rangle, |1\rangle\}$. We assume that the qubits have identical interactions with their own individual environments, and neglect the mutual interaction between qubits. The dynamics of the ith qubit $1 \leqslant i \leqslant N$ is then described by a completely positive trace-preserving map (or channel) $\mathcal{E}_i$, so that $\rho = \mathcal{E}_i \rho_0$, while the evolution of the N-qubit system is given by the composition of all N individual maps: $\rho \equiv \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_N \rho_0$. The assumption of mutually independent and identical environments is well justified whenever the separation between the particles is much larger than a typical length associated with the environment (like a resonant wavelength of an electromagnetic environment), so that collective effects do not need to be taken into account.

The phase damping (or dephasing) channel (D) represents the situation in which there is loss of coherence with probability p, but without any energy exchange. It is defined as

$$\varepsilon_i^D \rho_i = (1-p)\rho_i + p(|0\rangle\langle0|\rho_i|0\rangle\langle0| + |1\rangle\langle1|\rho_i|1\rangle\langle1|).$$
(6)

The application of this map in the computational basis $\{|0\rangle, |1\rangle\}$ clearly does not affect the populations, but all the original coherences $|i\rangle\langle j|$ get reduced by a decay factor $(1-p)$.

The amplitude-damping channel (AD) is given, in the Born-Markov approximation, via its Kraus representation, as

$$\mathcal{E}_i^{\mathrm{AD}} \rho_i = E_0 \rho_i E_0^\dagger + E_1 \rho_i E_1^\dagger,$$
(7)

with $E_0 \equiv |0\rangle\langle0| + \sqrt{1-p}|1\rangle\langle1|$ and $E_1 \equiv \sqrt{p}|0\rangle\langle1|$. Here $|1\rangle$ stands for the excited state and $|0\rangle$ for the ground state of a two-level system. In this case, the population of the excited state is reduced by a factor $(1-p)$, while the population of the ground state is equal to the initial population plus the contribution coming from the upper state, which is equal to p multiplied by the initial population of that state. The coherences, on the other hand, are reduced by the factor $\sqrt{1-p}$.

The parameter p in channels (6) and (7) is a convenient parametrization of time: $p = 0$ refers to the initial time $t = 0$ and $p = 1$ refers to the asymptotic $t \to \infty$ limit. In the Markovian approximation, and for the amplitude channel, one has $p = 1 - \exp(-\gamma t)$, where γ is the decay rate of the excited state. This corresponds to the well-known Weisskopf-Wigner approximation. It should be remarked that these two maps, when acting on GHZ state (1) and W-like states (2), (3), (4), and (5), do not create new coherences, but for the amplitude-damping channel new diagonal terms (populations) may be created.

III. ENTANGLEMENT MEASURES

To investigate the entanglement features of the states here considered and the relation of these features with different communication tasks, we compare different quantifiers of

entanglement. In particular, we evaluate explicitly the dynamics of the negativity [20] and concurrence [23], measures of bipartite entanglement in a given bipartition of a multipartite state, and of the Meyer-Wallach measure of global entanglement (MW) [24]. We also compare the negativity, concurrence and the MW measure with the generalized concurrence [25], a measure of genuine multipartite entanglement that had its dynamics in the W state (2) numerically calculated in [8].

The negativity, given a bipartition $\{A\} : \{B\}$ of a multipartite state, is defined as the absolute value of the sum of the negative eigenvalues of the partially transposed (PT) density matrix $\rho_{AB}^{T_A}$ [26], which can be defined in terms of the trace norm $\|\rho_{AB}^{T_A}\|$, the sum of moduli of the eigenvalues of $\rho_{AB}^{T_A}$, as [20]

$$\mathcal{N}(\rho) = \frac{\|\rho_{AB}^{T_A}\| - 1}{2}. \tag{8}$$

In general, the negativity fails to quantify entanglement of some entangled states (those with positive partial transposition) in dimensions higher than six [26]. However, for the states considered here, under the influence of the two maps considered in the previous section, dephasing and amplitude damping, the negativity vanishes only when the state is a separable one. So, in these cases, the negativity brings all the relevant information about the separability in bipartitions of the states (i.e., null negativity means separability in the corresponding partition). In particular, as we show in the following section, the bipartite entanglement problem for W-like states (2), (3), (4), and (5) can always be reduced to two qubits, a situation where the negativity is an unambiguous entanglement quantifier.

The concurrence [23] for a given bipartition $\{A\} : \{B\}$ of a multipartite pure state $|\psi\rangle$ is

$$C(|\psi\rangle) = \sqrt{2(1 - \mathrm{tr}\rho_A^2)}, \tag{9}$$

being $\rho_A = \mathrm{tr}_B(\rho_{AB})$. This measure can be extended over the mixed states $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ by virtue of a convex roof construction [27]

$$C(\rho) = \inf_{\{p_i, |\Psi_i\rangle\}} \sum_i p_i C(|\Psi_i\rangle), \tag{10}$$

an optimization that can be analytically evaluated for two-qubit states, the minimum in this case is obtained by

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{11}$$

with λ_i the eigenvalues, λ_1 denoting the largest among them, of the matrix $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where the conjugation is taken with respect to the computational basis $\{|0\rangle, |1\rangle\}$.

For a pure state $|\psi\rangle$ of N qubits and a partition $\mathcal{A}|\mathcal{B}$, the corresponding nonlocal information $S_{\mathcal{A}|\mathcal{B}}$ between $\mathcal{A}$ and $\mathcal{B}$ is distributed among different kinds of contributions [28]

$$S_{\mathcal{A}|\mathcal{B}} = \sum_{k=2}^{N} I_{i_1 i_2 \ldots i_k}, \tag{12}$$

where the sum is taken over all possible combinations of indices such that $i_1, i_2, \ldots, i_k$ are not in the same set $\mathcal{A}$ or $\mathcal{B}$, and $I_{i_1 i_2 \ldots i_k}$ is some appropriate measure of nonlocal information

shared among all the k qubits. The quantity $S_{\mathcal{A}|\mathcal{B}}$ can be taken [28] as the mutual information

$$S_{\mathcal{A}|\mathcal{B}} = S_{\mathcal{A}} + S_{\mathcal{B}} - S_{\mathcal{A}\mathcal{B}}, \tag{13}$$

with $S_{\mathcal{Y}} = (1 - \mathrm{tr}\rho_{\mathcal{Y}}^2)$ being the linear entropy. Since $|\psi\rangle$ is pure, $S_{\mathcal{A}\mathcal{B}} = 0$, the entropies of the two parts $\mathcal{A}$ and $\mathcal{B}$ are identical, and therefore $S_{\mathcal{A}|\mathcal{B}} = 2(1 - \mathrm{tr}\rho_{\mathcal{A}}^2)$. For example, for two-qubit pure states $I_{12} = \tau_{12}$, where τ_{12} is the square of the concurrence [23] and is called the 2-tangle. For three-qubit pure states, the corresponding nonlocal information between 1 and 23 can be written as $S_{1|23} = I_{12} + I_{13} + I_{123}$ with $I_{12} = \tau_{12}$, $I_{13} = \tau_{13}$, and $I_{123} = \tau_{123}$, where the 3-tangle τ_{123} has been shown to be a well-defined measure of genuine three-qubit entanglement [29].

The MW measure for entanglement [24] for a pure state $|\psi\rangle$ of N qubits can be expressed as [30]

$$E_{\mathrm{MW}}(|\psi\rangle) = \frac{1}{N} \sum_{i=1}^{N} 2(1 - \mathrm{tr}\rho_i^2), \tag{14}$$

an average over the entanglement (square of the concurrence) of each qubit with the rest of the system. From Eqs. (12) and (13) we can see that the MW measure can be distributed among different kinds of nonlocal information

$$E_{\mathrm{MW}}(|\psi\rangle) = \frac{1}{N} \left(2 \sum_{i_1 < i_2} I_{i_1 i_2} + \cdots + N \sum_{i_1 < i_2 \cdots < i_N} I_{i_1 i_2 \ldots i_N} \right). \tag{15}$$

From this last equation it is clear that the MW measure depends on the different forms of quantum correlations present in the entangled state and, in fact, it was originally described [24] as a global entanglement measure. However, the MW measure precludes a detailed knowledge of the different quantum correlations $I_{i_1 i_2 \ldots i_k}$ in the system. For example, it cannot distinguish fully entangled states from states that, in spite of being entangled, are separable in some of their subsystems [31].

The generalized concurrence [25] is a natural extension of the concurrence (9) for multipartite N-qubit states and can be expressed as [8]

$$C_N = 2^{1-(N/2)} \sqrt{(2^N - 2) - \sum_\alpha \mathrm{tr}\rho_\alpha^2}, \tag{16}$$

where α labels all possible reduced density matrices. Note that the expression inside the square-root is nothing more than the sum of the concurrences in all possible bipartitions. The generalized concurrence can detect real multipartite correlations and it allows one to compare the degree of entanglement of multipartite systems with different numbers of constituents, as opposed to the MW measure.

In the following section, we show that, for the states (2), (3), (4), and (5), any convex (bipartite or multipartite) entanglement quantifier that does not increase under LOCC's, in any given partition, can be expressed in terms of that of a considerably smaller subsystem consisting only of those qubits lying on the boundary of the partition. No optimization on the full system's parameter space is required throughout.

IV. ENTANGLEMENT DYNAMICS OF W STATES

Following the same line of thought as in [10], we decompose the W state as a set of two-qubit unitary transformations acting on a separable state; the unitary transformations that act inside the parts are local unitary operations with respect to the partition and therefore do not change its entanglement.

To generate the states $|W_N\rangle$ and $|W_N^0\rangle$ using two-qubit operations and starting with a separable state, we need to apply a series of operations of the type

$$U_{i,j}^{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{\mu-\nu}{\mu}} & \sqrt{\frac{\nu}{\mu}} & 0 \\ 0 & \sqrt{\frac{\nu}{\mu}} & \sqrt{\frac{\mu-\nu}{\mu}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (17)$$

where i, j are the qubits affected by the transformation, while μ and ν are parameters defining the transformation. States (2) and (4) can be written as [see Fig. 1(a)]

$$|W_N\rangle = U|10\cdots00\rangle,$$
$$|W_N^0\rangle = U(\alpha|10\cdots00\rangle + \beta|00\cdots00\rangle), \quad (18)$$

with $U = \prod_{i=0}^{N-2} U_{N-1-i,N-i}^{i+2,i+1}$. Different orderings of the operators $U_{i,j}^{\mu,\nu}$ are possible depending on the kind of partition we are interested in. This is explicitly shown in Eq. (22). Also the $|\widetilde{W}_N^0\rangle$ state can be decomposed in this form, but now the two-qubit transformations explicitly depend on the parameters a_i that define the states (3) and (5). This decomposition of the states is now used to calculate their entanglement dynamics.

A. Amplitude damping

The action of individual channels of this type on the states (2) yields

$$\rho_W^{\mathrm{AD}} = p|0_N\rangle\langle 0_N| + (1-p)|W_N\rangle\langle W_N|, \quad (19)$$

with $|0_N\rangle = |0\rangle^{\otimes N}$. This state can be written in terms of the unitary transformations (17) as

$$\rho_W^{\mathrm{AD}} = U[(p|00\rangle\langle 00| + (1-p)|10\rangle\langle 10|)$$
$$\otimes |0_{N-2}\rangle\langle 0_{N-2}|]U^\dagger. \quad (20)$$

The states $|W_N\rangle$ and $|W_N^0\rangle$ are invariant by permutations, so which qubits are in each part is irrelevant. For the least balanced bipartition $\{1\} : \{N-1\}$, only the unitary U_{12} acting on the boundary of this bipartition affects the entanglement since the unitary transformations acting inside the parts are local transformations with respect to this partition [see Fig. 1(b)]. So the entanglement in this case can be seen to be

$$E(\rho_W^{\mathrm{AD}}) = E[p|0_2\rangle\langle 0_2| + (1-p)|W_2(1)\rangle\langle W_2(1)|], \quad (21)$$

with $|W_2(k)\rangle = U_{1,2}^{N,N-k}|10\rangle = \sqrt{\frac{N-k}{N}}|01\rangle + \sqrt{\frac{k}{N}}|10\rangle$. The passage from (20) to (21) follows from the fact that the local addition of ancillas does not change the entanglement. This is similar to the strategy adopted in [10], where it was shown that the entanglement dynamics in a given partition of a graph state is the same as the one of the particles in the boundary of
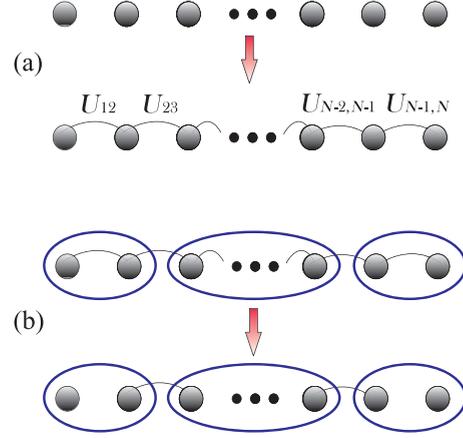

FIG. 1. (Color online) (a) Starting from a separable state the W state is constructed by applying a series of unitary transformations given by (17). (b) The entanglement corresponding to a given partition of the W state is equivalent to the entanglement of a subsystem consisting of only the qubits connected by the unitary transformations that cross the boundaries of the partition.

the partition. For a bipartition $\{k\} : \{N-k\}$ we can order the unitary transformations so that the W state is expressed as

$$|W_N\rangle = \prod_{j=k}^{2} \prod_{i=k}^{N-2} \left(U_{j,j-1}^{j,j-1} U_{i+1,i+2}^{N-i,N-i-1}\right)$$
$$\times U_{k,k+1}^{N,N-k}|0_1,\ldots,1_k,\ldots,0_N\rangle. \quad (22)$$

Following the same reasoning, the entanglement in this bipartition $\{k\} : \{N-k\}$ can be seen to be

$$E(\rho_W^{\mathrm{AD}}) = E(p|0_2\rangle\langle 0_2| + (1-p)|W_2(k)\rangle\langle W_2(k)|). \quad (23)$$

The same is valid for higher-order partitions. Depending on the way the entangled state is partitioned, the order of the operators $U_{i,j}^{\mu,\nu}$ must be properly chosen so as to factor out the unitary transformations that are irrelevant for the entanglement in the partition under consideration. For instance, suppose we want to calculate the entanglement in a tripartition $\{1\} : \{2\} : \{3,4\}$ of a four-qubit W state. If one write this state as $U_{12}U_{23}U_{34}$ applied to a separable state, then every single unitary transformation must be taken into account, as opposed to the alternative description $U'_{34}U'_{12}U'_{23}$ applied to another separable state, where the last unitary transformation U'_{34} can be neglected since it is local with respect to the part $\{3,4\}$.

This formalism is valid for any convex and monotonic entanglement measure. Similar expressions can be found to (3), (4), and (5) evolving under the AD channel.

B. Dephasing

The action of individual channels of this type on the state (2) leaves the diagonal elements untouched, while the coherences gain a factor $(1-p)^2$. The evolved state $|W_N\rangle$ can be written as

$$\rho_W^D = (1-p')\sum_{k=1}^{N}|k_N\rangle\langle k_N| + p'|W_N\rangle\langle W_N|, \quad (24)$$

with $p' = (1-p)^2$ and $|k_N\rangle = (1/\sqrt{N})|0,\ldots,1_k,\ldots,0\rangle$. Following the same recipe as for the AD channel, the entanglement in a bipartition $\{k\} : \{N-k\}$ of the dephased $|W_N\rangle$ is given by

$$E\big(\rho_W^D\big) = E\left((1-p')\sum_{k=1}^{N}|k_N\rangle\langle k_N| + p'|W_2(k)\rangle\langle W_2(k)|\right.$$

$$\left.\otimes |0_{N-2}\rangle\langle 0_{N-2}|\right). \tag{25}$$

We apply to the last $N-2$ qubits in the state (25) a separable POVM [32] with two elements given by $A_1 = \sum_{k=1}^{N-2}|k_{N-2}\rangle\langle k_{N-2}|$ with $A_1^\dagger A_1 = \sum_{k=1}^{N-2}|k_{N-2}\rangle\langle k_{N-2}| < I_{N-2}$ and $A_2 = I_{N-2} - \sum_{k=1}^{N-2}|k_{N-2}\rangle\langle k_{N-2}|$ with $A_2^\dagger A_2 = A_2$. The completeness relation $A_1^\dagger A_1 + A_2^\dagger A_2 = I_{N-2}$ is satisfied. The post measurement state associated with A_1 is

$$\frac{A_1\rho_W^D A_1^\dagger}{\mathrm{tr}\big(A_1\rho_W^D A_1^\dagger\big)} = \sum_{k=1}^{N}|k_N\rangle\langle k_N|, \tag{26}$$

and the state associated with A_2 is

$$\frac{A_2\rho_W^D A_2^\dagger}{\mathrm{tr}\big(A_2\rho_W^D A_2^\dagger\big)} = |W_2(k)\rangle\langle W_2(k)| \otimes |0_{N-2}\rangle\langle 0_{N-2}|. \tag{27}$$

The POVM is separable and cannot raise the entanglement, which leads to a lower bound for the entanglement

$$E\big(\rho_W^D\big) \geqslant p' E(|W_2(k)\rangle\langle W_2(k)|).$$

The convexity of entanglement, directly applied to (24), yields an upper bound for the entanglement

$$E\big(\rho_W^D\big) \leqslant p' E(|W_2(k)\rangle\langle W_2(k)|). \tag{28}$$

The lower and upper bounds coincide and therefore give an exact quantification of the entanglement. Similar expressions can be found for the states (3), (4), and (5) undergoing a dephasing process.

V. BIPARTITE VERSUS GLOBAL ENTANGLEMENT DYNAMICS

Using as a measure of nonlocal information the n-tangle [33], we show in the Appendix that W-like states (2), (3), (4), and (5) are completely characterized by 2-tangles

$$E_{\mathrm{MW}}\big(|\widetilde{W}_N^0\rangle\big) = \frac{2}{N}\sum_{i_1<i_2}\tau\big(|\widetilde{W}_N^0\rangle\big)_{i_1 i_2}, \tag{29}$$

all the other genuine multipartite entanglement tangles being null. Since the decoherence acting on the states is local, which cannot increase the entanglement, the global entanglement dynamics is completely specified by the 2-tangle dynamics. This is a fortunate circumstance since the 2-tangle can be analytically evaluated even for mixed states, being directly related to the concurrence, while higher-order tangles require, in general, involved optimizations.

For the two-qubit state ρ_{ij} obtained after tracing over $N-2$ qubits, the 2-tangle for the AD channel is given by $\tau_{ij}^{AD} = |\alpha|^4|a_i|^2|a_j|^2(1-p)^2$, while for the dephasing channel we get

$\tau_{ij}^D = |\alpha|^4|a_i|^2|a_j|^2(1-p)^4$. The MW measure, in this case the sum over all 2-tangles, is therefore given, respectively, by

$$E_{\mathrm{MW}}(\rho^{AD}) = \frac{2}{N}(1-p)^2\sum_{i<j}^{N}|\alpha|^4|a_i|^2|a_j|^2$$

$$= (1-p)^2 E_{\mathrm{MW}}\big(|\widetilde{W}_N^0\rangle\big), \tag{30}$$

and

$$E_{\mathrm{MW}}(\rho^{PD}) = (1-p)^4 E_{\mathrm{MW}}\big(|\widetilde{W}_N^0\rangle\big). \tag{31}$$

In both cases, the MW entanglement decay is size independent. This is a generalization of the result obtained in [9], restricted to the state (2).

One should note that our method, as proposed in Sec. IV, allows a considerable simplification in the evaluation of the entanglement corresponding to a given bipartition. Indeed, in the usual approach, the negativity associated to some bipartition is calculated by partially transposing the density matrix. Given the symmetry of the states (2) and (4) under permutation of the qubits, we can restrict these partial transpositions to the $N/2$ first qubits ($N/2 + 1$ for N odd). Under the partial transposition on the bipartition $\{k\} : \{N-k\}$ of the evolved density matrix, $2k(N-k)$ coherences, originally in the subspace of one excitation, go to the subspace of two excitations. With this remark we could, in principle, calculate analytically the eigenvalues of the partially transposed density matrix as functions of p, N, and k, which would involve, in general, finding the roots of a N^2th order polynomial. However, as we have shown in the last section, the bipartite entanglement can be calculated through a much simpler way, using a two-qubit density matrix.

Using the concurrence as the quantifier of bipartite entanglement we see even more clearly the power of our method. For multipartite mixed states, the calculation of the concurrence would be attempted by a brute-force optimization approach given by Eq. (11), but since, in this particular case we treat here, the boundary qubits are just two, the use of our method allows us to perform the calculation with no optimization at all, for an explicit formula for the concurrence exists for arbitrary two-qubit systems.

For the state (2) undergoing dephasing and amplitude damping, the negativities in a bipartition $\{k\} : \{N-k\}$ can be shown to be given, respectively, by

$$\mathcal{N}_D(p,N,k) = \frac{(1-p)^2}{N}\sqrt{k(N-k)}, \tag{32}$$

$$\mathcal{N}_{\mathrm{AD}}(p,N,k) = -\frac{p}{2} + \frac{1}{2N}\sqrt{N^2 p^2 + 4k(N-k)(1-p)^2}. \tag{33}$$

The concurrences for bipartitions $\{k\} : \{N-k\}$ of the state (2) undergoing dephasing and amplitude damping are given, respectively, by

$$C_D(p,N,k) = 2\frac{(1-p)^2}{N}\sqrt{k(N-k)}, \tag{34}$$

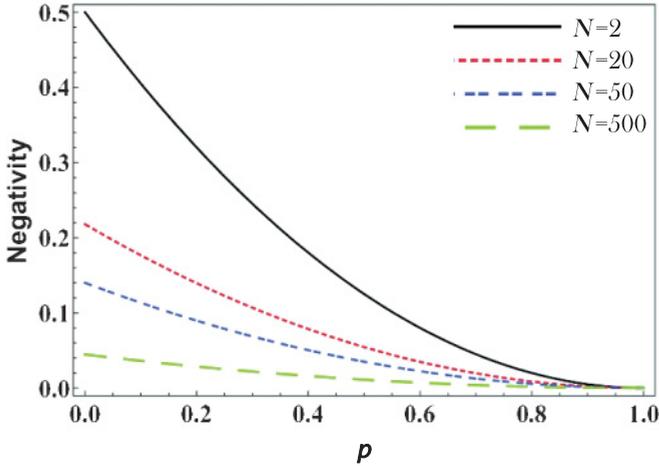$$C_{\mathrm{AD}}(p,N,k) = 2\frac{(1-p)}{N}\sqrt{k(N-k)}, \tag{35}$$

FIG. 2. (Color online) The negativity corresponding to the least balanced bipartition as a function of p and $N = 2$, 20, 50, and 500, for the W state undergoing individual dephasing. The initial entanglement decreases as the number of qubits increases, but the decay factor is independent of N and given by $(1 - p)^2$.
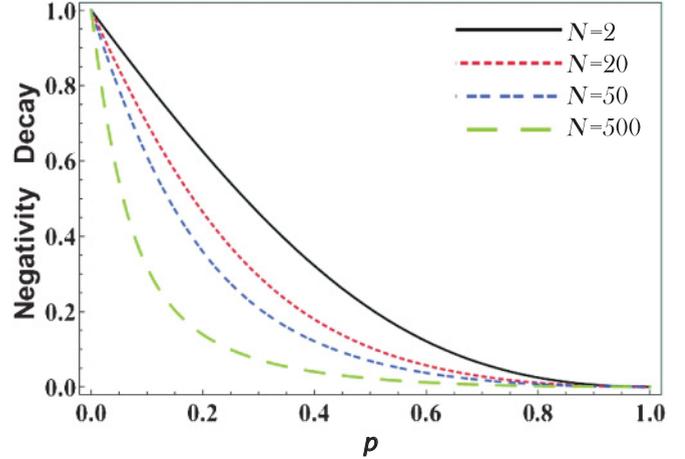

FIG. 3. (Color online) The decay of the negativity corresponding to the least balanced bipartition as a function of p for $N = 2$, 20, 50, and 500, for the W state undergoing individual amplitude damping. Even though the global entanglement decay in the state is size independent, the entanglement decay in this bipartition, for a fixed value of p, scales as $1/\sqrt{N}$, for N sufficiently large.

Setting $p = 0$ we can analyze the bipartite entanglement of the initial W state. For $k = N/2$ the entanglement is independent of N and maximal; in fact, the balanced bipartition can always be written in terms of effective qubits as a maximally entangled two-qubit state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. For other k's the initial bipartite entanglement of the W state depends on N; for a given value of k the W state behaves like an effective two-qubit entangled state of the form $|\widetilde{\Psi^+}\rangle = (\sqrt{\frac{N-k}{N}}|01\rangle + \sqrt{\frac{k}{N}}|10\rangle)$, for which the initial entanglement has an intrinsic dependence on N. For a fixed k, the larger N is the more this bipartition approximates a separable one. This can be viewed in Fig. 2, where we plot the negativity of the least balanced bipartition as a function of time in the state (2) undergoing dephasing. The initial entanglement decreases with the number of qubits.

From Eqs. (32), (33), (34), and (35) we can see that the W state does not undergo finite-time disentanglement. However, if we are interested not in the disappearance of the initial entanglement, but in the survival of a significant fraction of it, either to be directly used or to be distilled without an excessively large overhead in resources, we need to look at its decay [7], that is, the behavior of the ratios $\frac{\mathcal{N}(p,N,k)}{\mathcal{N}(0,N,K)} = \epsilon$ and $\frac{C(p,N,k)}{C(0,N,K)} = \delta$ as a function of p.

Under dephasing the decay of negativity and concurrence of any bipartition is expressed by the factor

$$\epsilon_D = \delta_D = (1 - p)^2. \tag{36}$$

For the amplitude damping the concurrence decay for any bipartition is size independent and given by

$$\delta_{AD} = (1 - p). \tag{37}$$

while the negativity decay is given by

$$\epsilon_{AD}(p,N,k) = \frac{-Np + \sqrt{N^2 p^2 + 4k(N-k)(1-p)^2}}{2\sqrt{k(N-k)}}, \tag{38}$$

which is independent of N for $k = N/2$. However, the more unbalanced is the bipartition, the faster is its decay. The idea is clearly illustrated in Fig. 3 for the least balanced bipartition. For $k = 1$ and any $p \neq 0$, it is easy to see that

$$\epsilon_{AD}(p,N,1) \propto \frac{1}{\sqrt{N}} \quad \text{when} \quad N \gg \left\{1, \frac{(1-p)^2}{p^2}\right\}, \tag{39}$$

which contrasts with the behavior of the GHZ states under the same channel, where for a fixed value of p the negativity decays exponentially with N [7].

The size independence for the decay factor for the concurrence in any bipartition of a W state (2) is also reflected in a size independence for the decay of the generalized concurrence. This can be easily seen through Eq. (16), since the decay factors, expressed in Eqs. (36) and (37), are independent of the bipartition. This is the analytical proof of the result obtained by numerical methods in [8].

Under dephasing the bipartite entanglement decay is independent of N and k, the same being true for the global entanglement [8,9]. The W state is extremely robust under the dephasing channel. Under the action of the amplitude damping channel, the bipartite entanglement of W states is much more robust than that of GHZ states, although it can still depend intrinsically on the number of particles N. This is an interesting and unexpected behavior since the global entanglement decay, as quantified by the MW measure and the generalized concurrence, is independent of N. The same conclusions are valid for the states (3), (4), and (5): Under dephasing and amplitude damping the global entanglement decay is size independent while the negativity decay can depend on N for the AD channel.

The different behavior of the global and bipartite quantifiers is now shown to have direct implications on the robustness of two distinct communication tasks.

VI. TELEPORTING AND SPLITTING QUANTUM INFORMATION

Entangled states are resources for important protocols for quantum communication. Let us consider now the effect of decoherence on two related tasks, which we show to be associated with different kinds of entanglement.

We consider first the teleportation [1] of an unknown quantum state $|\psi\rangle = \sum_i^d a_i |i\rangle$ of dimension d. To perfectly realize the teleportation, Alice and Bob need to share a maximally entangled bipartite state $|\Phi^+\rangle_{AB} = (1/\sqrt{d})\sum_i^d |i\rangle_A |i\rangle_B$. If they share a nonmaximally entangled state they will be able to realize an imperfect teleportation. The quality of the teleportation is quantified by the fidelity, defined as $f = \langle\psi|\rho|\psi\rangle$, where ρ is the state obtained after teleportation. The maximum achievable fidelity $f_{\max}$ for a given bipartition of the state used as a channel is bounded by the negativity $\mathcal{N}$ associated to this bipartition [1,20]

$$f_{\max} \leqslant \frac{2+2\mathcal{N}}{d+1}. \tag{40}$$

The splitting of quantum information [21] is a generalization of the teleportation protocol. Alice has an unknown qubit $|\Psi_0\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ she would like to send to N other parts, a many-Bobs system, in such a way that the N Bobs must cooperate in order that just one of them extracts the quantum information. This is the best one can have, in view of the no-cloning theorem [34], which implies that only one copy of $|\Psi_0\rangle$ can be received. Alice and the other N parts share an entangled state and proceed in a very similar way as in the usual quantum teleportation [1]. First, Alice teleports the qubit to the N Bobs, a usual teleportation related to the negativity in the bipartition. We show now that the second part of the protocol, when the N parts cooperate to extract the information, is related to the global entanglement of the shared state.

We consider, under decoherence, two distinct protocols to split quantum information, which use GHZ and a W-like state as resources. Note from (40) that to teleport a qubit we need to use a channel that has at least $\mathcal{N} = 1/2$. This is true for any bipartition of a GHZ state [7], but as we have shown this is no longer true for any biparton of a W state (2). The only bipartition of a W state with $\mathcal{N} = 1/2$ is the balanced bipartition. To use a W state as the channel for the splitting of information between the N Bobs, we need to use the balanced partition in a W state with $2N$ qubits. The more unbalanced is the bipartition the worse is the initial teleportation fidelity and the faster will be the fidelity decay under decoherence. Another possible choice for a perfect teleportation, the one we consider here, is to use an asymmetric W_A state with $N+1$ qubits of the form

$$\left|W_A^{N+1}\right\rangle = \frac{1}{\sqrt{2}}[|0\rangle|W_N\rangle + |1\rangle|0\rangle^{\otimes N}], \tag{41}$$

which has the required $\mathcal{N} = 1/2$ in the bipartition $\{1\} : \{N\}$.

In both the GHZ and W_A protocols, Alice measures the two qubits in her possession in a Bell basis, communicating the classical outcomes to the Bobs (Fig. 4). In the GHZ protocol,
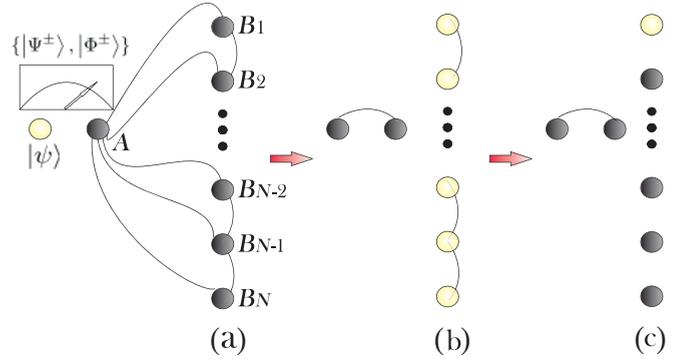

FIG. 4. (Color online) Scheme for the teletransport and the splitting of quantum information involving $N+1$ parts. The edges represent the entanglement between the qubits, which is created or consumed along the protocol. (a) Alice measure her qubit and the qubit $|\psi\rangle$ she wants to teletransport (split) in a Bell basis. (b) The teletransported state is encoded in a N-qubit state. (c) The N parties must cooperate so that only one of them has at the end the teleported state $|\psi\rangle$.

after the Bell measurement performed by Alice, the N Bobs state will be given by one of the four states

$$
\begin{aligned}
|\widetilde{\Phi}^{+,-}\rangle &= \alpha|0\rangle^{\otimes N} \pm \beta|1\rangle^{\otimes N}, \\
|\widetilde{\Psi}^{+,-}\rangle &= \alpha|1\rangle^{\otimes N} \pm \beta|0\rangle^{\otimes N}.
\end{aligned} \tag{42}
$$

After they decide who among them will be the receiver of the state $|\Psi_0\rangle$, the nonreceivers measure their qubits in the basis of eigenvectors of X, where X is the σ_x Pauli matrix, and communicate their outcomes to the receiver. In possession of all the classical measurement outcomes, the receiver can recover the teleported state since the state in his possession is

$$Z^M X^{a_2} Z^{a_1}|\Psi_0\rangle, \tag{43}$$

M being the number of Bobs measurements that returned $|-\rangle$, and the indices a_1 and a_2 are related to the measurement made by Alice, so that $\{|\Phi^+\rangle,|\Psi^+\rangle\} \Rightarrow a_1 = 0$, $\{|\Phi^-\rangle,|\Psi^-\rangle\} \Rightarrow a_1 = 1$, $\{|\Phi^+\rangle,|\Phi^-\rangle\} \Rightarrow a_2 = 0$, and $\{|\Psi^+\rangle,|\Psi^-\rangle\} \Rightarrow a_2 = 1$.

Using the W_A state as the resource, after Alice's measurement the four possible N-qubit state outcomes are

$$
\begin{aligned}
|\widetilde{\Phi}^{+,-}\rangle &= \alpha|0\rangle^{\otimes N} \pm \beta|W_N\rangle, \\
|\widetilde{\Psi}^{+,-}\rangle &= \alpha|W_N\rangle \pm \beta|0\rangle^{\otimes N}.
\end{aligned} \tag{44}
$$

Note this is nothing more than the usual teleportation, with the information encoded in effective qubits. Note also that the shared state is in the form (4). In the decodification part, all the N Bobs need to meet and then apply a global operation on their qubits so that

$$|W_N\rangle \to |1\cdots 00\rangle, \quad |0\cdots 00\rangle \to |0\cdots 00\rangle, \tag{45}$$

in such a way the information is now only contained in one of the qubits [21]. The above transformation is realized by the inverse of the unitary transformation displayed in Eq. (18).

Under amplitude damping of each qubit in the resource states GHZ and W_A, the respective fidelities in the teleportation
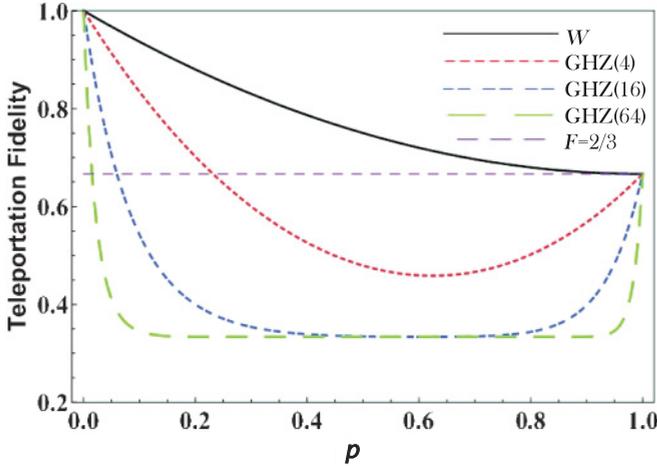
FIG. 5. (Color online) Teleportation fidelities for protocols that use as resources GHZ and W_A states under the AD channel. The W_A protocol is the best possible under AD and throughout its evolution always leads to fidelities above the classical threshold, while the GHZ protocol leads to fidelities below the classical threshold.

part of the protocol, averaged over all the possible input states $|\Psi_0\rangle$, can be shown to be

$$\overline{f^T}_{\text{GHZ}} = \tfrac{1}{6}[2 + (1 - p)^{N-1}(2 - p)$$
$$+ 2(1 - p)^{N/2} + p^{N-1}(1 + p)] \qquad (46)$$

$$\overline{f^T}_{W_A} = \tfrac{1}{3}(3 - 2p + p^2). \qquad (47)$$

Considering the complete splitting protocol, the teleportation followed by the decodification, the fidelities are

$$\overline{f}^D_{\text{GHZ}} = \frac{1}{3}[2 - p(1 - p) + (1 - p)^{N/2}]$$
$$\overline{f^D}_{W_A} = 1 - \frac{p}{3}. \qquad (48)$$

For the asymptotic separable state ($p = 1$) we see that the average classical fidelity 2/3 is recovered in all the previous expressions. In the W_A state, the entanglement corresponding to the partition between Alice and the N Bobs decays as given in Eq. (38) with $k = N/2$. The negativity in the bipartition used for the teleportation in the state (41) under AD is such that f_{max}, as given by Eq. (40), is equal to $\overline{f^T}_{W_A}$. This implies that not only is the protocol robust, but also it is the best possible. The same is not true for the GHZ protocol under decoherence since $\overline{f^T}_{\text{GHZ}}$ decays exponentially with N and along the evolution this fidelity can be under the classical limit, in spite of the fact that the negativity is null only in the asymptotic limit $p = 1$. The teleportation fidelities are plotted in Fig. 5.

The resource state used for the decodification is given by (44), which is a W-like state (4), which has a global entanglement decay independent of N, while its bipartite entanglement, quantified by the negativity, depends on N. The fact that the fidelity of the whole protocol is size independent is a clear indication that the decodification part of the protocol depends on the global entanglement, which should be expected since this decodification depends on all parts in a symmetric way. Therefore, the comparison of the decays of the entanglement and of the fidelity indicates on which kind

of entanglement the task relies. In the decoherent scenario, similar but nonidentical protocols can rely on different types of entanglement, which can imply distinct robustness of these distinct tasks. For the teleportation and splitting of quantum information, protocols based on W-like states are clearly much more robust than GHZ-based protocols.

VII. CONCLUSION

The characterization of entanglement for multipartite systems is a complex endeavor in view of the many possible partitionings of the state and the possibility of having, for a given partition, many possible quantifiers. Even for two qubits, two widely used quantifiers, the concurrence and the negativity, do not lead to consistent results when comparing the entanglement of two states. This motivates the search for criteria that would associate each quantifier with a different physical task.

This problem becomes even more pressing when considering the effects of the environment. Different quantifiers may display quite distinct behaviors under decoherence. This motivates a natural question: Can these different dynamics be associated to the robustness of inequivalent communication tasks?

In this paper we have shown that two of those quantifiers, proposed for estimating global and bipartite entanglements, respectively, do behave quite distinctively under decoherence, and that this divergent behavior is directly related to the robustness of two inequivalent communication tasks.

We have made a detailed comparison between the use of GHZ or W states as resources for quantum communication in a noisy environment, by applying a new technique that allows a reduction of the W-state entanglement dynamics to that of a two-qubit system. This technique could be possibly applied to other classes of entangled states under different kinds of environment.

Our approach suggests that the eventual ambiguities in the definition of entanglement quantifiers should be resolved by associating each quantifiers with a definite physical task.

APPENDIX: W STATES HAVE ONLY 2-TANGLE

To show that (29) is valid for the W-like states $|\widetilde{W}_N^0\rangle$, we employ the MW measure in the form proposed in [24]

$$E_{\text{MW}}(|\psi\rangle) = \frac{4}{N} \sum_{i=1}^{N} D\big(l_i(0)|\psi\rangle, l_i(1)|\psi\rangle\big). \qquad (A1)$$

Here $l_i(b)$ is a $(C^2)^{\otimes N} \to (C^2)^{\otimes N-1}$ map

$$l_i(b)|k_1, \ldots, k_N\rangle = \delta_{bk_i}|k_1, \ldots, \widehat{k_i}, \ldots, k_N\rangle, \qquad (A2)$$

where $\widehat{k_i}$ means the absence of the term k_i and $|k_1,\ldots,k_N\rangle$ is the computational basis with $k_i = \{0,1\}$. So when $l_i(b)$ acts on an N-dimensional vector state $|\psi\rangle$, it generates an $(N-1)$-dimensional vector state $|\psi^{i,b}\rangle = \sum_{k=1}^{2^{N-1}} u_k^{i,b}|k\rangle$, being $k = (k_1,\ldots,k_{N-1})$. The quantity $D(l_i(0)|\psi\rangle, l_i(1)|\psi\rangle)$ is a distance defined by

$$D\big(l_i(0)|\psi\rangle, l_i(1)|\psi\rangle\big) = \sum_{x<y} \big|u_x^{i,0}u_y^{i,1} - u_y^{i,0}u_x^{i,1}\big|^2. \quad \text{(A3)}$$

The application of the operator l_i to the $|\widetilde{W}_N^0\rangle$ gives

$$l_i(0)\big|\widetilde{W}_N^0\big\rangle = \sum_{k=1}^{2^{N-1}} u_k^{i,0}|k\rangle = \beta|0\rangle + \sum_{j\neq i}^{N} \alpha a_j|j\rangle,$$

$$l_i(1)\big|\widetilde{W}_N^0\big\rangle = \sum_{k=1}^{2^{N-1}} u_k^{i,1}|k\rangle = \alpha a_i|0\rangle. \quad \text{(A4)}$$

Therefore

$$D\big(l_i(0)\big|\widetilde{W}_N^0\big\rangle, l_i(1)\big|\widetilde{W}_N^0\big\rangle\big) = \sum_{x<y} \big|u_x^{i,0}u_y^{i,1} - u_y^{i,0}u_x^{i,1}\big|^2$$

$$= \sum_{j\neq i}^{N} \big|u_j^{i,0}u_0^{i,1}\big|^2 = \sum_{j\neq i}^{N} |\alpha|^4|a_j a_i|^2, \quad \text{(A5)}$$

and the MW entanglement measure for $|\widetilde{W}_N^0\rangle$ is thus

$$E_{\text{MW}}\big(\big|\widetilde{W}_N^0\big\rangle\big) = \frac{4}{N}\sum_{i=1}^{N}\sum_{j\neq i}^{N} |\alpha|^4|a_j a_i|^2$$

$$= \frac{8|\alpha|^4}{N}\sum_{i<j}^{N} |a_j a_i|^2. \quad \text{(A6)}$$

Tracing out, in the state $|\widetilde{W}_N^0\rangle$, every qubit but qubits i and j, the two-qubit density matrix ρ_{ij} becomes

$$\begin{pmatrix} |\beta|^2 + |\alpha|^2(1-|a_i|^2-|a_j|^2) & \alpha\beta^*a_i^* & \alpha\beta^*a_j^* & 0 \\ \alpha^*\beta a_i & |\alpha|^2|a_i|^2 & |\alpha|^2 a_i^* a_j & 0 \\ \alpha^*\beta a_j & |\alpha|^2 a_i a_j^* & |\alpha|^2|a_j|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(A7)}$$

The concurrence of this state is given by $C_{ij} = 2|\alpha|^2|a_i a_j|$ and so the 2-tangle is $\tau_{ij} = C_{ij}^2 = 4|\alpha|^4|a_i a_j|^2$. From this result and from (A6) it can be easily seen that

$$E_{\text{MW}}\big(\big|\widetilde{W}_N^0\big\rangle\big) = \frac{2}{N}\sum_{i<j}^{N} \tau_{ij}. \quad \text{(A8)}$$

Therefore the global entanglement of state $|\widetilde{W}_N^0\rangle$ is completely characterized by its 2-tangles.

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