Quantum physics frontiers
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Heraklion Crete, July 24-28, 2017

Instituto de Física UFRJ

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

Luiz Davidovich
Instituto de Física - Universidade Federal do Rio de Janeiro

THE ONASSIS FOUNDATION
SCIENCE LECTURE SERIES

Outline of the lectures

These two lectures will focus on recent developments in quantum metrology. The main questions to be answered are:

(i) What are the ultimate precision limits in the estimation of parameters, according to classical mechanics and quantum mechanics?
(ii) Are there fundamental limits? Is quantum mechanics helpful in reaching better precision?
(iii) How to cope with the deleterious effects of noise?

Our discussion is restricted to local quantum metrology: in this case, one is not interested in an optimal globally-valid estimation strategy, valid for any value of the parameter to be estimated, but one wants instead to estimate a parameter confined to some small range. The techniques to be developed are useful, for instance, for estimating parameters that undergo small changes around a known value, like sensing phase changes in gravitational-wave detectors; or yet if one has some prior (eventually rough) knowledge about the value of the parameter.

Summary of the lectures

The lectures will be organized as follows:

LECTURE 1. General introduction: parameter estimation and classical bounds on precision. The Cramér-Rao bound and the Fisher information. Extension of Cramér-Rao bound and Fisher information to quantum mechanics. Quantum Fisher information for pure states. The role of entanglement. Application to optical and atomic interferometry

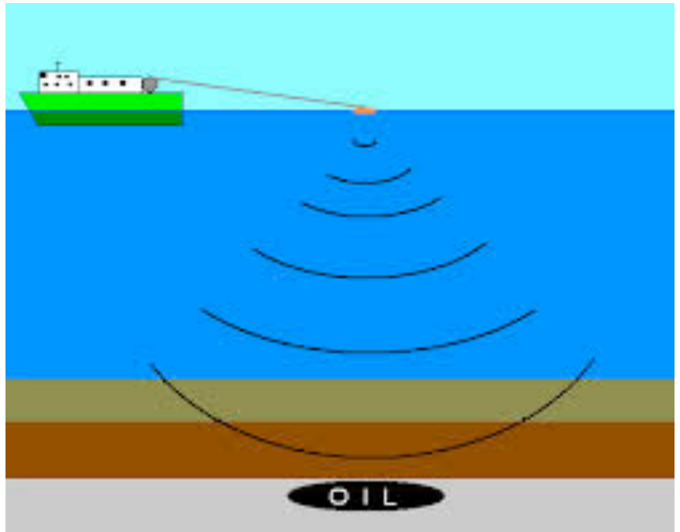
LECTURE 2. Noisy quantum-enhanced metrology: General framework for evaluating the ultimate precision limit in the estimation of parameters. Application to optical interferometers and force estimation. Quantum metrology and the energy-time uncertainty relation. Generalization to open systems. Application to atomic decay.

For more details, see Lectures at College de France (2016):
http://www.if.ufrj.br/~ldavid/eng/show_arquivos.php?Id=5

I.1 - General introduction: parameter estimation and classical limits on precision

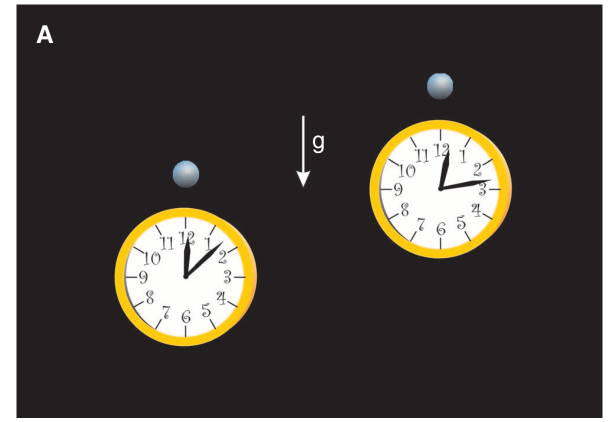
Parameter estimation

Depth of an oil well



Time duration of a process

Transition frequency



$$\Delta h = 33\,\mathrm{cm}$$

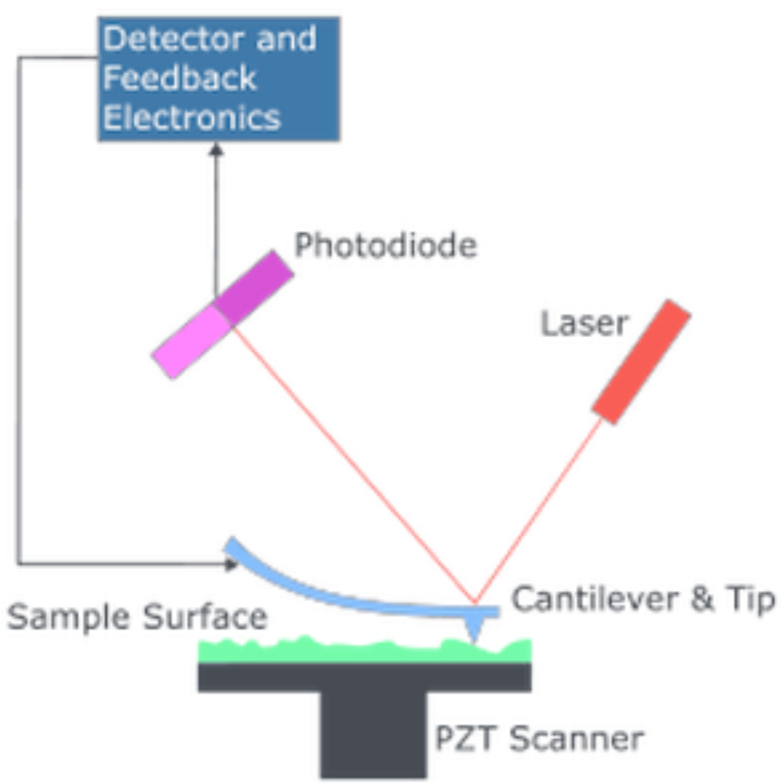
$$\frac{\Delta f}{f} = (4.1 \pm 1.6) \times 10^{-17}$$

Optical Clocks and Relativity

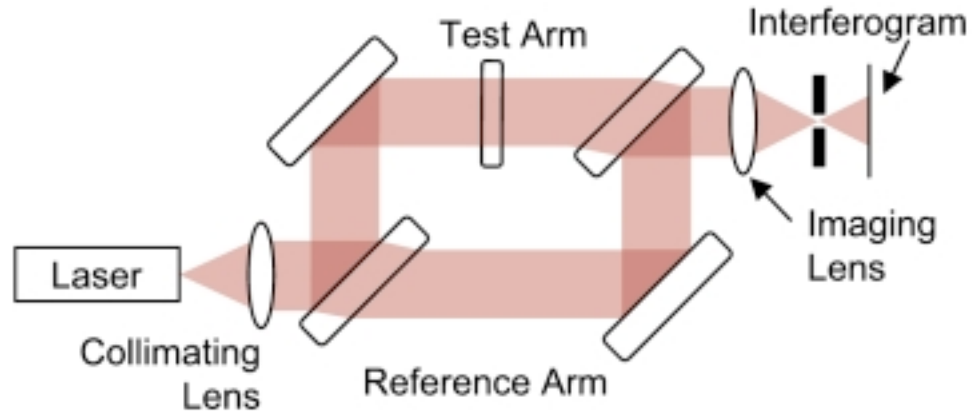
C. W. Chou,* D. B. Hume, T. Rosenband, D. J. Wineland

24 SEPTEMBER 2010 VOL 329 **SCIENCE**

Weak forces or small displacements



Phase displacements in interferometers



Laser Interferometer Gravitational Wave Observatory

nature photonics
LETTERS
PUBLISHED ONLINE: 21 JULY 2013 | DOI: 10.1038/NPHOTON.2013.177

Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light

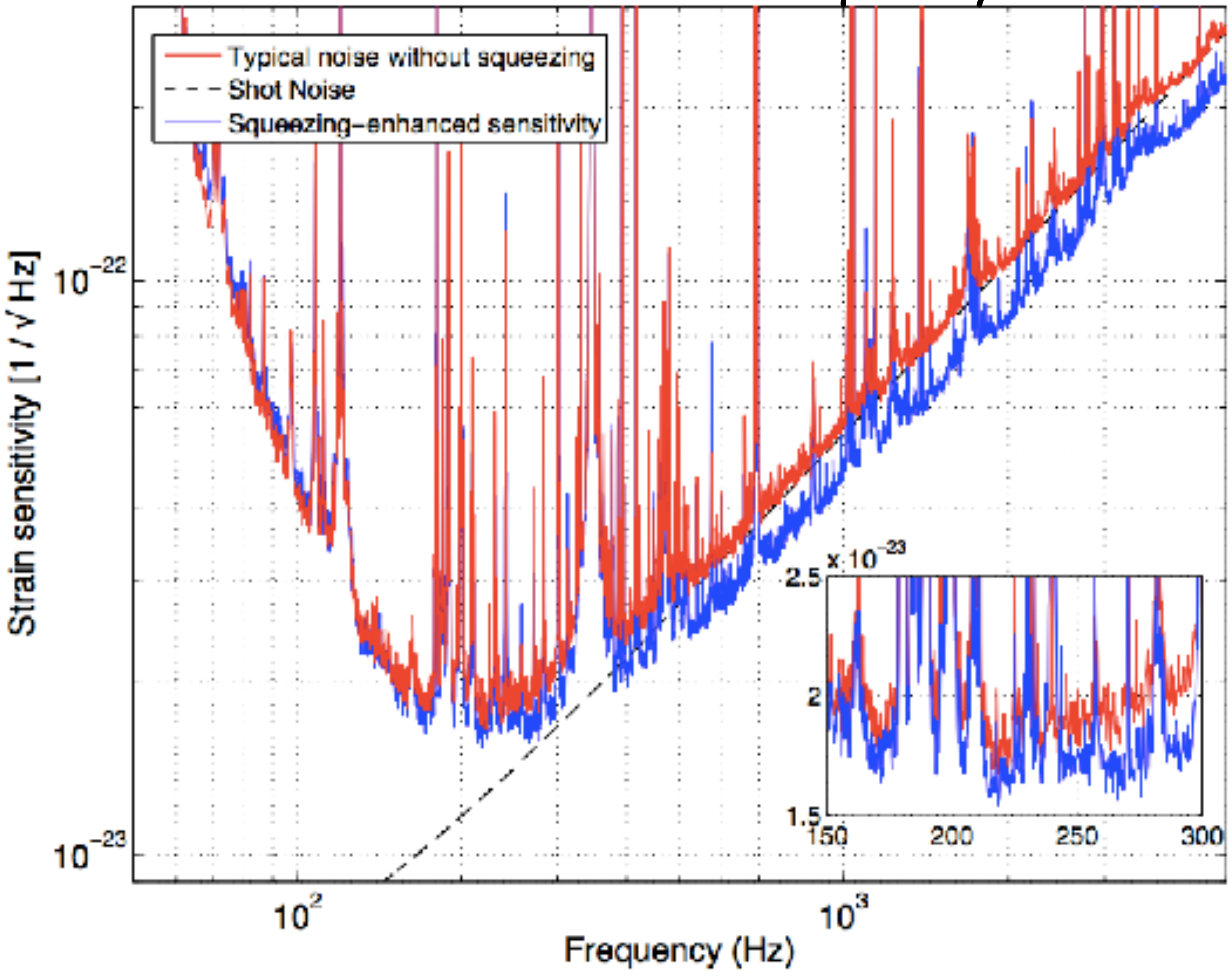
The LIGO Scientific Collaboration*

High-precision interferometry: Advanced LIGO

Differential displacement sensitivity $\simeq 10^{-19}\,\mathrm{m}$

Relative change in distance $\simeq 3\times 10^{-23}$

Up to 2.15dB improvement in sensitivity in the shot-noise- limited frequency band



Experiments: Parameter estimation beyond classical physics in the XXI century

Phase resolution

Letter

Nature Photonics **4**, 357 - 360 (2010)
Published online: 4 April 2010 | doi:10.1038/nphoton.2010.39

Experimental quantum-enhanced estimation of a lossy phase shift

M. Kacprowicz[1], R. Demkowicz-Dobrzański[1,2], W. Wasilewski[2], K. Banaszek[1,2] & I. A. Walmsley[3]

NATURE PHOTONICS | LETTER

Entanglement-enhanced measurement of a completely unknown optical phase

G. Y. Xiang, B. L. Higgins, D. W. Berry, H. M. Wiseman & G. J. Pryde

Letters to Nature

Nature **429**, 161-164 (13 May 2004) | doi:10.1038/nature02493; Received 22 December 2003; Accepted 16 March 2004

Super-resolving phase measurements with a multiphoton entangled state

M. W. Mitchell, J. S. Lundeen & A. M. Steinberg

1. Department of Physics, University of Toronto, 60 St George Street, Toronto, Ontario M5S 1A7, Canada

New Journal of Physics
The open access journal at the forefront of physics

Beating the standard quantum limit: phase super-sensitivity of *N*-photon interferometers

Ryo Okamoto[1,2,5], Holger F Hofmann[3], Tomohisa Nagata[1], Jeremy L O'Brien[4], Keiji Sasaki[1] and Shigeki Takeuchi[1,2]

Science 4 May 2007:
Vol. 316 no. 5825 pp. 726–729
DOI: 10.1126/science.1138007

REPORT

Beating the Standard Quantum Limit with Four-Entangled Photons

Tomohisa Nagata[1], Ryo Okamoto[1,2], Jeremy L. O'Brien[3,4], Keiji Sasaki[1], Shigeki Takeuchi[1,2,*]

Experiments: Parameter estimation beyond classical physics in the XXI century

10960–10965 | PNAS | July 7, 2009 | vol. 106 | no. 27

Mesoscopic atomic entanglement for precision measurements beyond the standard quantum limit

J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjærgaard, and E. S. Polzik[1]

Danish National Research Foundation Center for Quantum Optics, The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Atomic clocks

Letter

Nature 443, 316-319 (21 September 2006) | doi:10.1038/nature05101; Received 5 May 2006; Accepted 18 July 2006

'Designer atoms' for quantum metrology

C. F. Roos[1,2], M. Chwalla[1], K. Kim[1], M. Riebe[1] & R. Blatt[1,2]

Science 4 June 2004:
Vol. 304 no. 5676 pp. 1476–1478
DOI: 10.1126/science.1097576

REPORT

Toward Heisenberg-Limited Spectroscopy with Multiparticle Entangled States

D. Leibfried[1], M. D. Barrett[1], T. Schaetz, J. Britton, J. Chiaverini, W. M. Itano, J. D. Jost, C. Langer, D. J. Wineland

Physics ABOUT BROWSE JOURNALISTS

Focus: Atomic Clock Beats the Quantum Limit

June 25, 2010 • *Phys. Rev. Focus* 25, 24

Researchers beat the quantum-mechanical fluctuations in an atomic clock by linking many atoms into an entangled quantum state and pushing the fluctuations into a realm that doesn't influence the time measurement.

New Journal of Physics
The open-access journal for physics

Entanglement-assisted atomic clock beyond the projection noise limit

Anne Louchet-Chauvet[1], Jürgen Appel, Jelmer J Renema, Daniel Oblak, Niels Kjaergaard[2] and Eugene S Polzik[3]

Implementation of Cavity Squeezing of a Collective Atomic Spin

Ian D. Leroux, Monika H. Schleier-Smith, and Vladan Vuletić
Phys. Rev. Lett. **104**, 073602 – Published 17 February 2010; Erratum Phys. Rev. Lett. **106**, 129902 (2011)

Experiments: Parameter estimation beyond classical physics in the XXI century

Magnetometers

NATURE | LETTER

Interaction-based quantum metrology showing scaling beyond the Heisenberg limit

M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell & M. W. Mitchell

NATURE | LETTER

日本語要約

Nature **510**, 376–380 (19 June 2014)

Measurement of the magnetic interaction between two bound electrons of two separate ions

Shlomi Kotler, Nitzan Akerman, Nir Navon, Yinnon Glickman & Roee Ozeri

Magnetic Sensitivity Beyond the Projection Noise Limit by Spin Squeezing

R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell
Phys. Rev. Lett. **109**, 253605 – Published 19 December 2012

Quantum Noise Limited and Entanglement-Assisted Magnetometry

W. Wasilewski, K. Jensen, H. Krauter, J. J. Renema, M. V. Balabas, and E. S. Polzik
Phys. Rev. Lett. **104**, 133601 – Published 31 March 2010; Erratum Phys. Rev. Lett. **104**, 209902 (2010)

Increasing Sensing Resolution with Error Correction

G. Arrad, Y. Vinkler, D. Aharonov, and A. Retzker
Phys. Rev. Lett. **112**, 150801 – Published 16 April 2014

Quantum Error Correction for Metrology

E. M. Kessler, I. Lovchinsky, A. O. Sushkov, and M. D. Lukin
Phys. Rev. Lett. **112**, 150802 – Published 16 April 2014

REPORTS

29 MAY 2009 VOL 324 **SCIENCE**

Magnetic Field Sensing Beyond the Standard Quantum Limit Using 10-Spin NOON States

Jonathan A. Jones,[1] Steven D. Karlen,[2] Joseph Fitzsimons,[2,3] Arzhang Ardavan,[1] Simon C. Benjamin,[2,4] G. Andrew D. Briggs,[2] John J. L. Morton[1,2]*

Parameter estimation and uncertainty relations

Parameter estimation and uncertainty relations

What is the meaning of

★ Time-energy uncertainty relation?

$$\Delta E \Delta T \geq \hbar / 2$$

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★ Number-phase uncertainty relation?

$$\Delta N \Delta \phi \geq \hbar / 2$$

Parameter estimation and uncertainty relations

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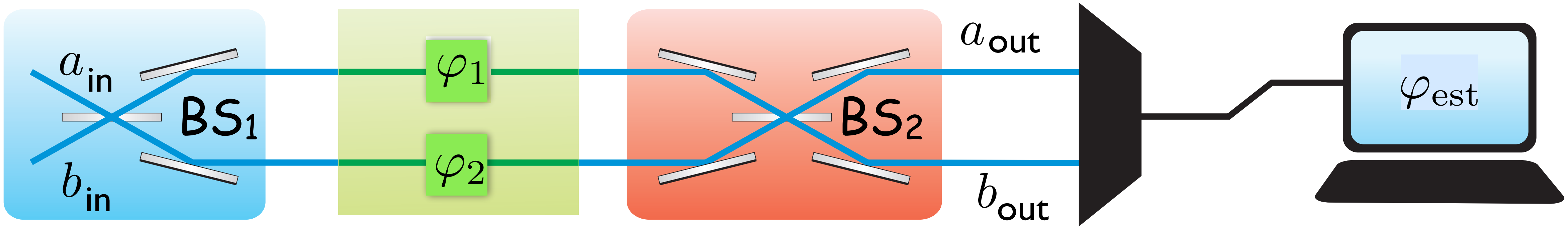
★ Number-phase uncertainty relation?

$$\Delta N \Delta \phi \geq \hbar / 2$$

We shall see that quantum parameter estimation allows to understand these relations in terms of uncertainties in the estimation of parameters: while Heisenberg uncertainty relations are associated with Hermitian operators, the theory of parameter estimation allows one to obtain uncertainty relations for parameters, like time or phase, with no need to associate them to suitable Hermitian operators.

An example: optical interferometry



Mach-Zender interferometer: a beam with complex amplitude $\mathbf{a}_{in}$ is split on a balanced beam splitter BS_1 and the two resulting beams acquire phases φ_1 and φ_2, interfering on the second beam splitter BS_2. The photon numbers $n_{a_{out}}$ and $n_{b_{out}}$ are measured at the output ports. One could also have two incident beams, with complex amplitudes $\mathbf{a}_{in}$ and $\mathbf{b}_{in}$.

The outgoing fields are related to the incoming ones through the transformation (note that $\mathbf{a}_{out}=\mathbf{a}_{in}$, $\mathbf{b}_{out}=\mathbf{b}_{in}$ when $\varphi_1 = \varphi_2 = 0$, since $[BS_1]X[BS_2]=1$) — replacing complex amplitudes are replaced by operators:

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{BS_2} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}}_{BS_1} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix}$$

Optical interferometry and Jordan-Schwinger transformation

PHYSICAL REVIEW A VOLUME 33, NUMBER 6 JUNE 1986

SU(2) and SU(1,1) interferometers

Bernard Yurke, Samuel L. McCall, and John R. Klauder
AT&T Bell Laboratories, Murray Hill, New Jersey 07974

(Received 30 October 1985)

This has the advantage of providing a unified formalism, which can also be applied to problems in atomic spectroscopy and magnetometry.

Let $\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{J}_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}), \quad \hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$

Then $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$ and $\hat{j}^2 = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2} + 1\right), \quad \hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$

so these operators obey the angular momentum algebra.

Transformations of operators $\hat{a}$ and $\hat{b}$ can be considered as rotations in spin space: $\hat{a}' = \hat{U}^\dagger \hat{a} \hat{U}$, $\hat{b}' = \hat{U}^\dagger \hat{b} \hat{U}$, with $\hat{U} = \exp(-i\theta \hat{J} \cdot \hat{n})$, where the unit vector $\hat{n}$ is along the axis of rotation, and with the correspondence:

BS1 $\rightarrow \hat{U} = \exp(-i\pi \hat{J}_x/2)$

BS2 $\rightarrow \hat{U} = \exp(i\pi \hat{J}_x/2)$

Phase delay $\rightarrow \hat{U} = \exp(-i\phi \hat{J}_z)$

$\phi = \varphi_2 - \varphi_1$

Angular momentum operators for optical interferometry

Corresponding transformation for the operators $\hat{J}_i$ (Heisenberg picture!):

$$\begin{pmatrix} \hat{J}_x^{out} \\ \hat{J}_y^{out} \\ \hat{J}_z^{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{J}_x^{in} \\ \hat{J}_y^{in} \\ \hat{J}_z^{in} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \hat{J}_x^{in} \\ \hat{J}_y^{in} \\ \hat{J}_z^{in} \end{pmatrix}$$

Therefore, Mach-Zender transformation amounts to a rotation around y axis of the angular momentum operators.

The state transforms as $$\left|\psi\right\rangle_{out} = e^{i\hat{J}_x\pi/2} e^{-i\hat{J}_z\varphi} e^{-i\hat{J}_x\pi/2} \left|\psi\right\rangle_{in}$$

Precision of phase estimation

From $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$, it is clear that $\hat{n}_a - \hat{n}_b = 2\hat{J}_z$.

On the other hand, the average of $\hat{J}_z$ in the output state is equal to the average of $\hat{J}_z^{\text{out}}$, given by the previous matrix expression, in the input state.

Therefore, $\langle \hat{J}_z \rangle_{\text{out}} = \cos\varphi \langle \hat{J}_z \rangle_{\text{in}} - \sin\varphi \langle \hat{J}_x \rangle_{\text{in}}$ while the variance is

$$\Delta^2 \hat{J}_z \Big|_{\text{out}} = \cos^2\varphi \, \Delta^2 \hat{J}_z \Big|_{\text{in}} + \sin^2\varphi \, \Delta^2 \hat{J}_x \Big|_{\text{in}} - 2\sin\varphi\cos\varphi \; \text{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\text{in}}$$

where the covariance cov is defined as

$$\text{cov}(\hat{J}_x, \hat{J}_z) = \tfrac{1}{2}\langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle - \langle \hat{J}_x \rangle \langle \hat{J}_z \rangle$$

The precision of estimation can now be quantified by the error propagation formula:

$$\Delta\varphi = \frac{\Delta \hat{J}_z \Big|_{\text{out}}}{\left| \frac{d\langle \hat{J}_z \rangle_{\text{out}}}{d\varphi} \right|}$$

where $\Delta\varphi = \sqrt{\Delta^2\varphi}$ is a standard deviation (same for $\Delta\hat{J}_z$).

Optical interferometry with Fock states

Consider that a Fock state $|N\rangle$ is injected in port a, so that $|\psi\rangle_{\text{in}} = |N\rangle_a |0\rangle_b$. Since

$$\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{J}_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}), \quad \hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$$

this initial state is an eingestate of $\hat{J}_z$ and $\hat{J}^2$: $\hat{J}_z |N,0\rangle = (N/2)|N,0\rangle$, $\hat{J}^2 |N,0\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right)|N,0\rangle$, so we may write $|N,0\rangle \to |j,j\rangle$. Also,

$$\langle \hat{J}_z \rangle_{\text{in}} = N/2,\ \langle \hat{J}_x \rangle_{\text{in}} = 0,\ \Delta^2 \hat{J}_z \Big|_{\text{in}} = 0\,,\ \Delta^2 \hat{J}_x \Big|_{\text{in}} = N/4,$$

and $\text{cov}(\hat{J}_x, \hat{J}_z)_{\text{in}} = 0\,.$

From $\langle \hat{J}_z \rangle_{\text{out}} = \cos\varphi \langle \hat{J}_z \rangle_{\text{in}} - \sin\varphi \langle \hat{J}_x \rangle_{\text{in}}$ and

$$\Delta^2 \hat{J}_z \Big|_{\text{out}} = \cos^2\varphi\ \Delta^2 \hat{J}_z \Big|_{\text{in}} + \sin^2\varphi\ \Delta^2 \hat{J}_x \Big|_{\text{in}} - 2\sin\varphi\cos\varphi\ \ \text{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\text{in}}$$

one gets

$$\Delta\varphi = \frac{\Delta \hat{J}_z \Big|_{\text{out}}}{\left|\frac{d\langle \hat{J}_z \rangle_{\text{out}}}{d\varphi}\right|} = \frac{\sqrt{N}|\sin\varphi|/2}{N|\sin\varphi|/2} = \frac{1}{\sqrt{N}},$$

which is the standard (or shot-noise limit) for optical interferometry.

Optical interferometry with Fock states

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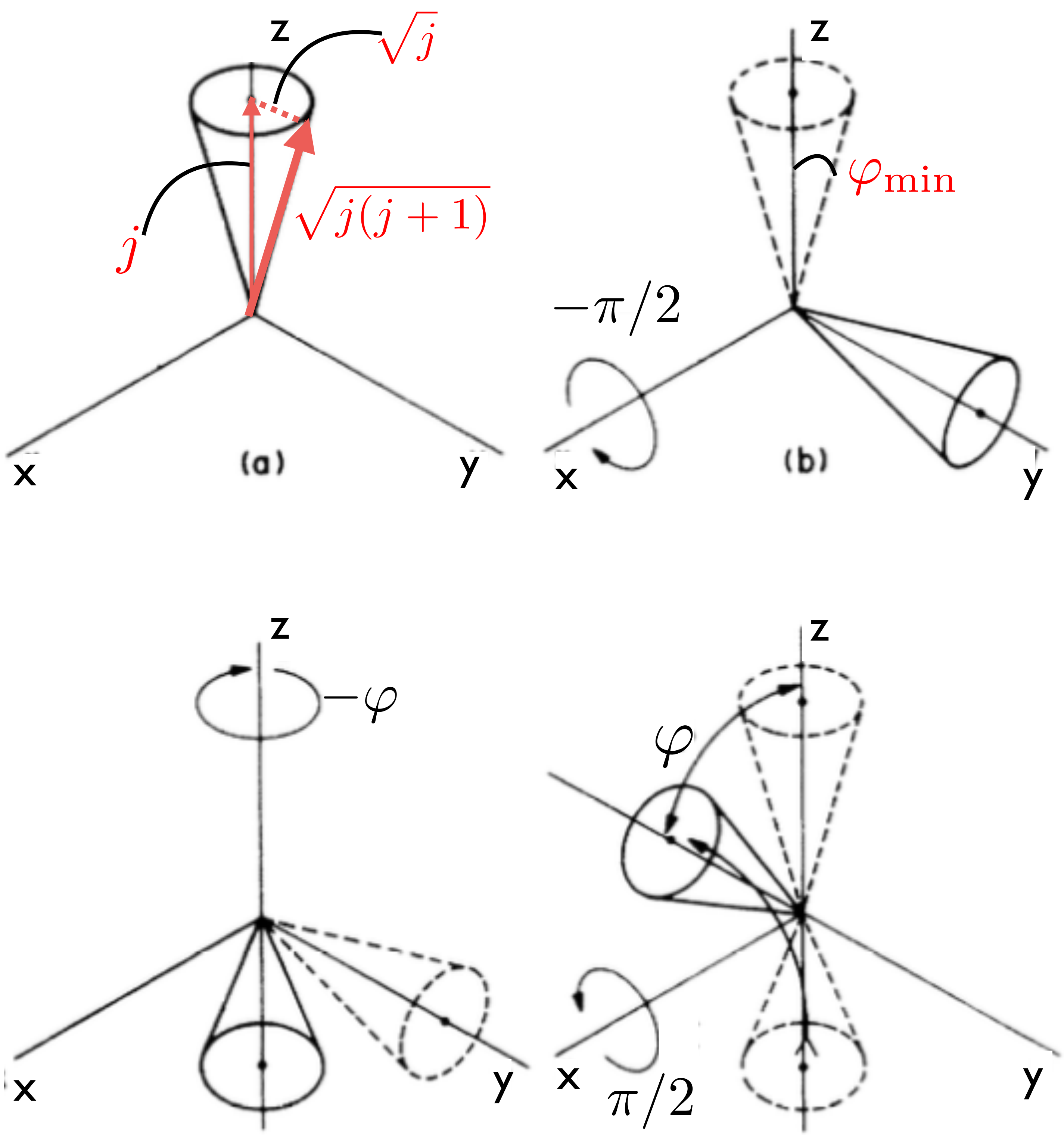
$$\Delta^2 \hat{J}_z\Big|_{\rm out} = \cos^2\varphi\ \Delta^2 \hat{J}_z\Big|_{\rm in} + \sin^2\varphi\ \Delta^2 \hat{J}_x\Big|_{\rm in} - 2\sin\varphi\cos\varphi\ \mathrm{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\rm in}$$

one gets

$$\Delta\varphi = \frac{\Delta \hat{J}_z\Big|_{\rm out}}{\left|\frac{d\langle \hat{J}_z\rangle_{\rm out}}{d\varphi}\right|} = \frac{\sqrt{N}|\sin\varphi|/2}{N|\sin\varphi|/2} = \frac{1}{\sqrt{N}},$$

which is the standard (or shot-noise limit) for optical interferometry.

Geometrical interpretation



- Length of side of the cone: $\sqrt{j(j+1)}$, with j=N/2
- Distance from apex to center of base: eigenvalue of $\hat{J}_z$ —> j=N/2
- Radius of the base of the cone:

$$\sqrt{j(j+1) - j^2} = \sqrt{j}$$

(a) Initial state

(b) Action of first beam splitter

(c) Phase delay

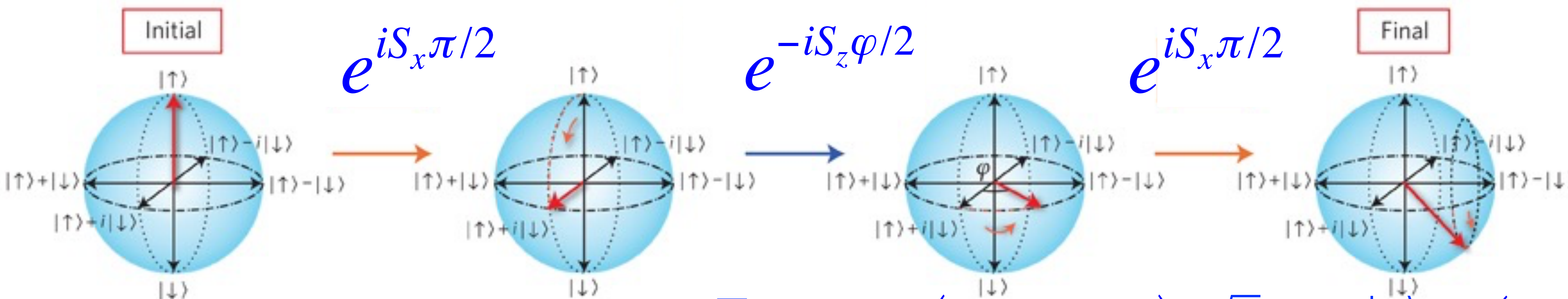
(d) Action of second beam splitter

Minimum detectable φ is of the order of

$$\varphi_{\min} \approx \frac{\sqrt{j}}{j} = \frac{1}{\sqrt{j}} \approx \frac{1}{\sqrt{N}}$$

Unified formalism for interferometers

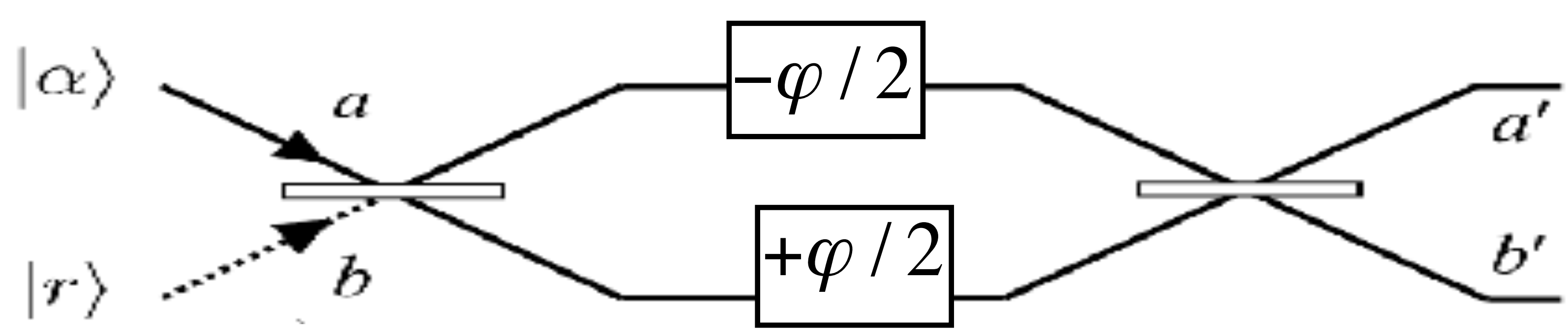
Ramsey interferometry



$$|e\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|e\rangle \rightarrow \left(|e\rangle + i|g\rangle\right)/\sqrt{2} \rightarrow |e\rangle \rightarrow \left(|e\rangle + ie^{i\varphi}|g\rangle\right)/\sqrt{2} \rightarrow -|e\rangle \sin\left(\varphi/2\right) + |g\rangle \cos\left(\varphi/2\right)$$

$$\langle J_z \rangle = 2(P_g - P_e) = \cos\phi$$



$$|\psi\rangle_{\text{out}} = e^{i\hat{J}_x\pi/2} e^{-i\hat{J}_z\varphi} e^{i\hat{J}_x\pi/2} |\psi\rangle_{\text{in}}$$

$$\varphi = \Delta\omega t$$

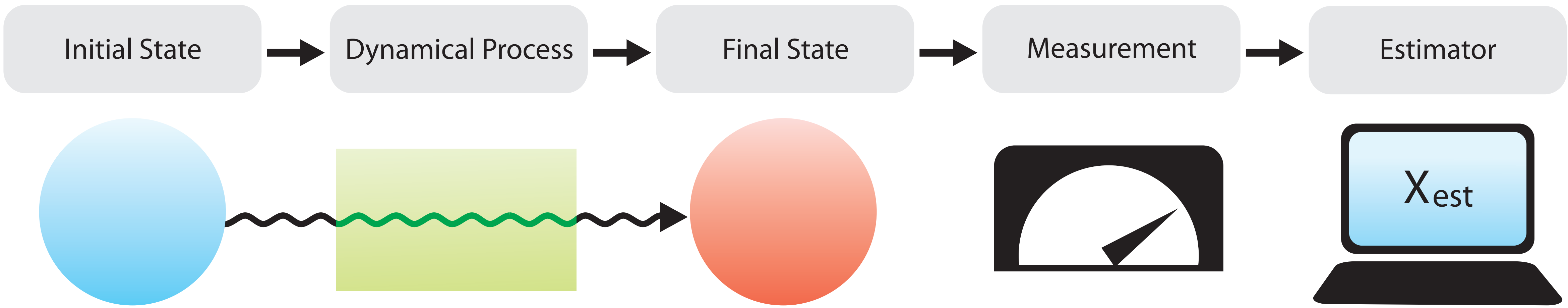
General estimation theory

1. What are the best possible measurements?

2. What are the best incoming states, in order to get better precision?

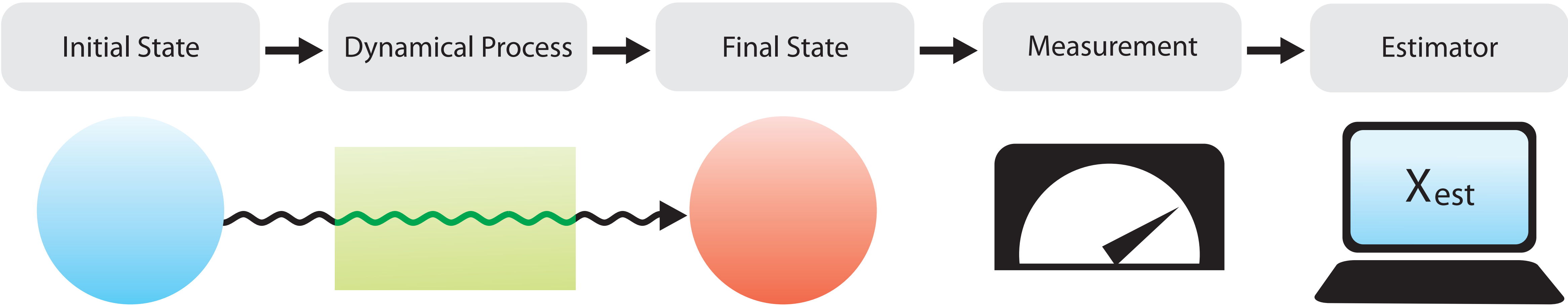
3. Is it possible to find *general* bounds and strategies for reaching them, which could be applied to many different systems?

Parameter estimation in classical and quantum physics



1. Prepare probe in suitable initial state
2. Send probe through process to be investigated
3. Choose suitable measurement
4. Associate each experimental result j with estimation

Parameter estimation in classical and quantum physics



1. Prepare probe in suitable initial state
2. Send probe through process to be investigated
3. Choose suitable measurement
4. Associate each experimental result j with estimation

$$\delta X \equiv \sqrt{\langle [X_{\rm est}(j) - X]^2 \rangle_j \Big|_{X=X_{\rm true}}} \rightarrow \text{ Merit quantifier}$$

$$\langle X_{\rm est} \rangle = X_{\rm true}, \; d\langle X_{\rm est}\rangle / dX \Big|_{X=X_{\rm true}} = 1 \rightarrow \text{ Unbiased estimator}$$

Then $\delta X^2 = \Delta^2 X = \left\langle [X_{\rm est} - \langle X_{\rm est}\rangle]^2 \right\rangle \rightarrow$ variance of X_{est} (average is taken over all experimental results)

Estimator depends only on the experimental data.

Classical parameter estimation


H. Cramér


C. R. Rao


R.A. Fisher

Cramér-Rao bound for unbiased estimators:

$$\Delta X \geq 1/\sqrt{N\,F(X)\big|_{X=X_{\text{true}}}}\,,\quad F(X) \equiv \sum_j P_j(X)\left(\frac{d\,\ln[P_j(X)]}{dX}\right)^2$$

Fisher information

$N \rightarrow$ Number of repetitions of the experiment

$P_j(X) \rightarrow$ probability of getting an experimental result j

or yet, for continuous measurements: $F(X) \equiv \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2$

where ξ are the measurement results

(Average over all experimental results)

Derivation of Cramér-Rao relation: See lectures by L. Davidovich at College de France, 2016:

http://www.if.ufrj.br/~ldavid/eng/show_arquivos.php?Id=5

Exercises

1. Show that

$$F(X) \equiv \int d\xi\, p(\xi|X) \left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 = \int d\xi \frac{1}{p(\xi|X)} \left[\frac{\partial p(\xi|X)}{\partial X}\right]^2$$

$$= 4\int d\xi \left[\frac{\partial \sqrt{p(\xi|X)}}{\partial X}\right]^2 = -\left\langle \frac{\partial^2}{\partial X^2} \ln p(\xi|X) \right\rangle$$

with similar expressions for a discrete set of measurements. For instance,

$$F(X) = \sum_k \left[\frac{d\sqrt{P_k(X)}}{dX}\right]^2$$

2. Let us consider several identical and independent measurements, so that the probability distribution is $p(\vec{\xi}|X) = p(\xi_1|X) \cdots p(\xi_N|X)$. Show that $F^{(N)}(X) = NF(X)$

Understanding the Fisher information (1)

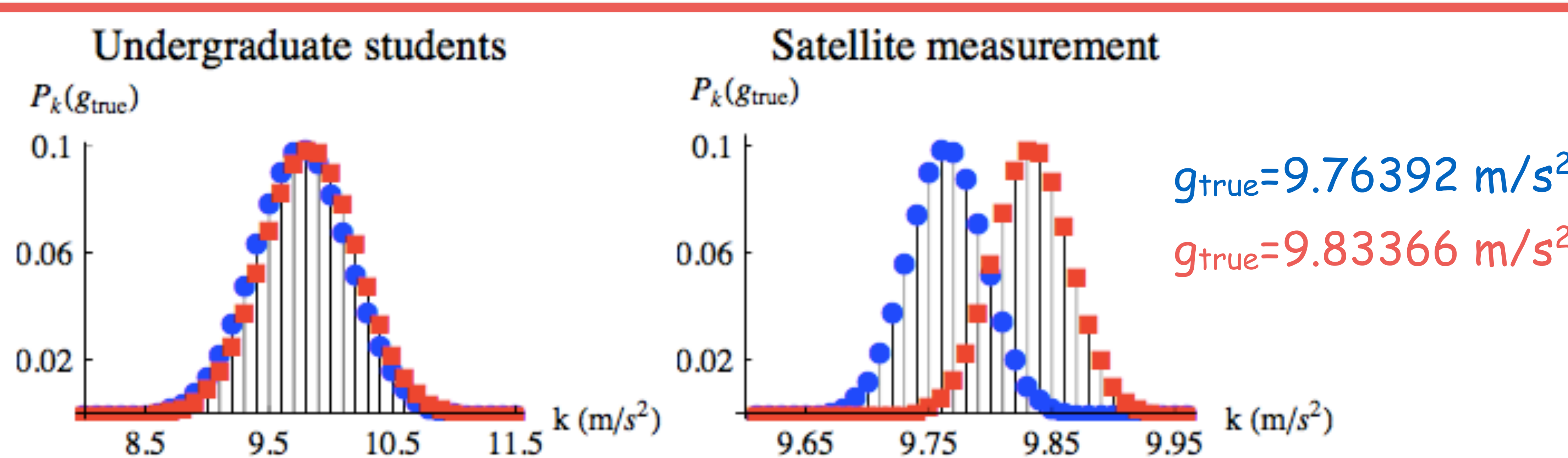
GEOPHYSICAL RESEARCH LETTERS, VOL. 40, 4279–4283, doi:10.1002/grl.50838, 2013

New ultrahigh-resolution picture of Earth's gravity field

Christian Hirt,[1] Sten Claessens,[1] Thomas Fecher,[2] Michael Kuhn,[1] Roland Pail,[2] and Moritz Rexer[1,2]

Márcio Mendes Taddei, Ph. D. thesis, Federal University of Rio de Janeiro, available at arXiv:1407.4343v1 [quant-ph]

The gravitational field is measured by undergraduate students, via an inclined-plane experiment, in two labs, situated at Huáscaran (Peruvian Andes) and the Artic Sea, so g_{true} is different in both cases. Their precision is one decimal place. The same measurement is made by higher-precision satellites, with one additional decimal place.



Values of $P_k(g_{\mathrm{true}})$ for a measurement of g in Huascarán, in the Andes (blue circles) and at the Arctic Sea (red squares). The distributions within each image are different because so is g_{true}. Measurement as made in a simple laboratory (left) is compared to that by higher-precision satellites (right).

Understanding the Fisher information (2)

The higher precision of the satellite experiments implies that it is easier to distinguish the true values of g from the P_k of these measurements. **Important question:** How much does the outcome distribution change by a change of the underlying true value of the parameter? I show now that the Fisher information is a measure of this change.

The distance between two probability distributions $\{P_k\}$ for a given set $\{k\}$ of outcomes, which differ because they belong to two different values x and x' of the parameter, can be defined by the **Hellinger expression** D_H:

$$D_H(x,x') = \sqrt{\frac{1}{2}\sum_k \left[\sqrt{P_k(x)} - \sqrt{P_k(x')}\right]^2}$$

Then,

$$D_H^2(x,x+dx) = \frac{1}{2}\sum_k \left[\sqrt{P_k(x+dx)} - \sqrt{P_k(x)}\right]^2 = \frac{1}{2}\sum_k \left[\frac{d}{dx}\sqrt{P_k(x)}\right]^2 dx^2$$

and

$$D_H^2(x,x+dx) = ds_H^2 = \frac{F(x)}{8}dx^2$$

F(X) as a measure of change of the probability distribution!

Understanding the Fisher information (3)

The expression for the Hellinger distance can be written in terms of the fidelity between the two distributions:

$$D_H(x,x') = \sqrt{\frac{1}{2}\sum_k \left[\sqrt{P_k(x)} - \sqrt{P_k(x')}\right]^2} = \sqrt{1-\sqrt{\Phi_H(x,x')}}$$

where

$$\Phi_H(x,x') = \left[\sum_k \sqrt{P_k(x)P_k(x')}\right]^2$$ (=1 for x=x')

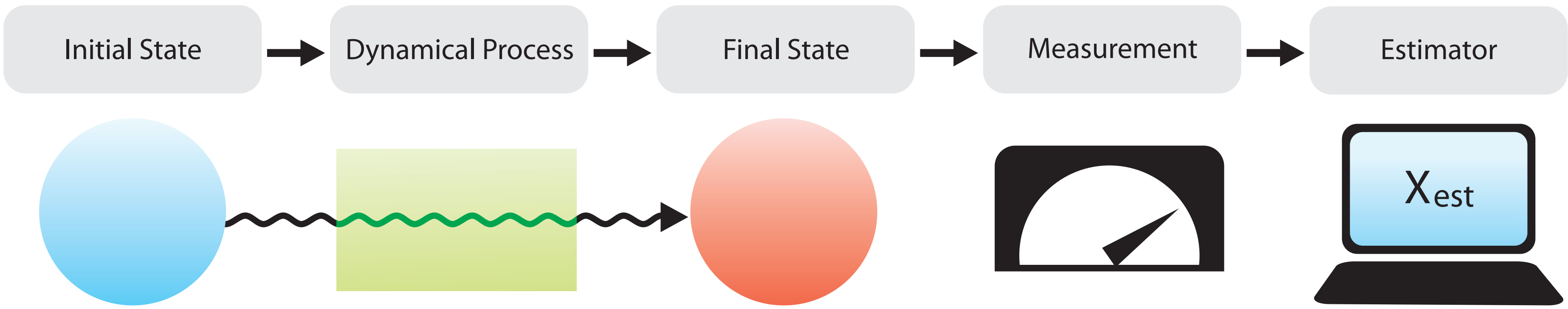
Therefore:

$$\Phi_H(x,x') = 1 - \frac{F(x)}{4}dx^2$$

$$\frac{\sqrt{F(x)}}{2} \rightarrow$$ Speed of change

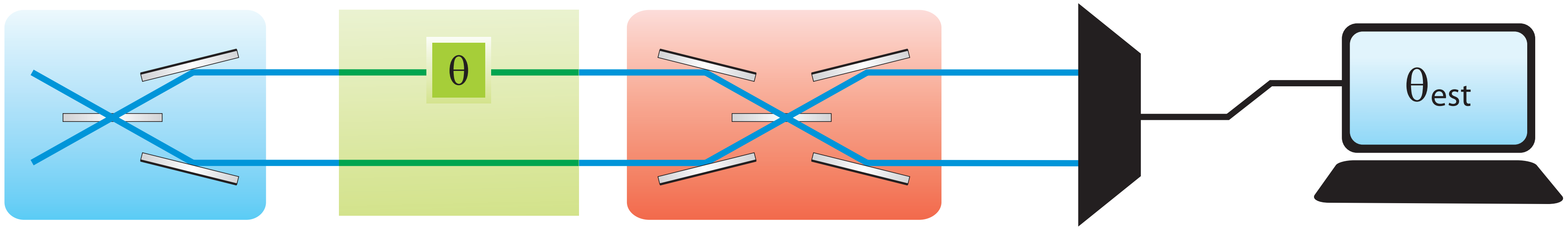
I.2 - Quantum parameter estimation

Quantum parameter estimation



The general idea is the same as before: one sends a probe through a parameter-dependent dynamical process and one measures the final state to determine the parameter. The precision in the determination of the parameter depends now on the distinguishability between quantum states corresponding to nearby values of the parameter.

Example: Optical interferometry



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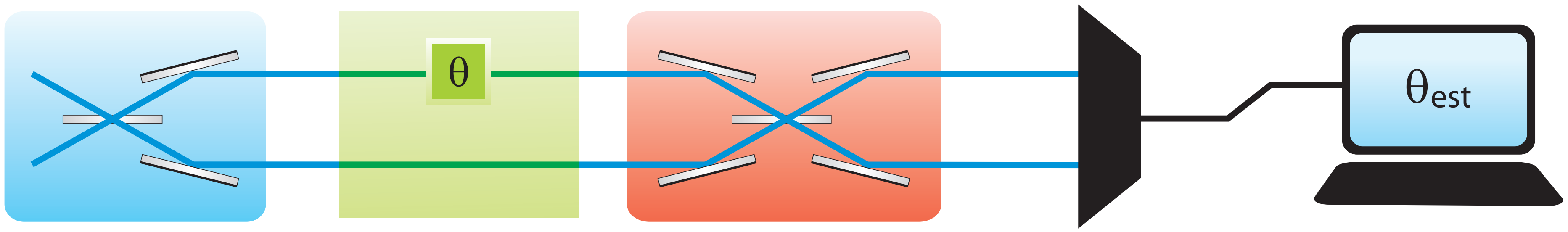

$$\left|\langle\alpha|\alpha e^{i\delta\theta}\rangle\right|^2 = \exp\left(-\left|\alpha\left(1-e^{i\delta\theta}\right)\right|^2\right)$$

$$\approx \exp\left[-\langle n\rangle(\delta\theta)^2\right] \Rightarrow \delta\theta \approx 1/\sqrt{\langle n\rangle}$$

→ Standard limit (shot noise)

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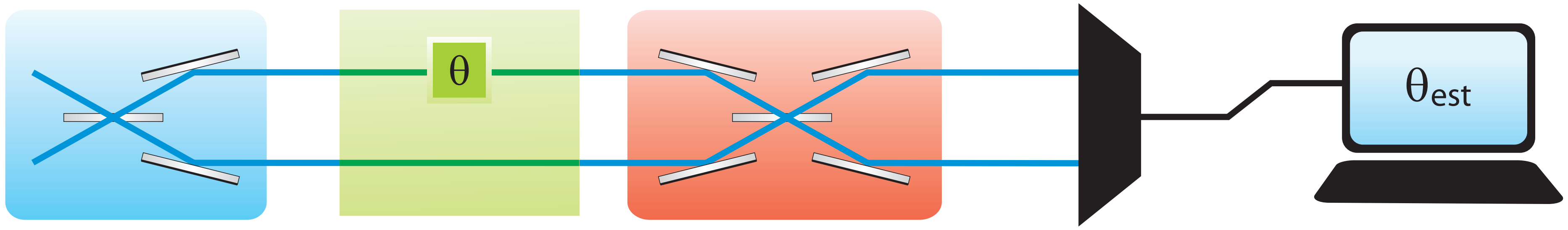

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$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2} \rightarrow |\psi(N,\theta)\rangle = \left(|N,0\rangle + e^{iN\theta}|0,N\rangle\right)/\sqrt{2}, \quad \left(\langle n\rangle = N\right)$$

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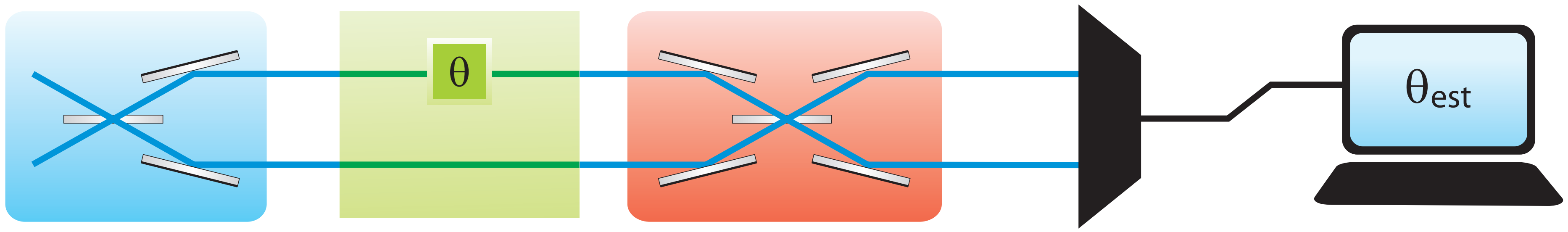

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$$\left|\langle\psi(N)|\psi(N,\delta\theta)\rangle\right|^2 = \cos^2(N\delta\theta/2) \Rightarrow \delta\theta \approx 1/N$$

$$\left[\cos^2(N\delta\theta/2) = 0 \Rightarrow \delta\theta = \pi/N\right]$$

HEISENBERG LIMIT — Precision is better, for the same amount of resources (average number of photons)!

Quantum Fisher Information

(Helstrom, Holevo, Braunstein and Caves)

$$F\left(X;\{\hat{E}_\xi\}\right) \equiv \int d\xi \; p(\xi \mid X)\left(\frac{d \ln\left[p(\xi \mid X)\right]}{dX}\right)^2$$

$$p(\xi \mid X) = \mathrm{Tr}\left[\hat{\rho}(X)\hat{E}_\xi\right]$$

$$\int d\xi \hat{E}_\xi = \hat{1}$$

POVM

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This corresponds to a given quantum measurement. Ultimate lower bound for $\langle(\Delta X_{\rm est})^2\rangle$: optimize over all quantum measurements so that

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Quantum Fisher Information

Quantum Fisher information for pure states

(See notes for derivation)

Initial state of the probe: $|\psi(0)\rangle$

Final X-dependent state: $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, $\hat{U}(X)$ unitary operator.

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Then (Helstrom 1976):

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

where

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$$\delta x \geq 1/2\sqrt{\nu\left\langle\Delta\hat{H}^2\right\rangle}$$

$\Rightarrow$ Should maximize the variance to get better precision!

Another expression for the quantum Fisher information

From

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

and $\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$

it follows that

$$\mathcal{F}_Q(X) = 4\left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right]$$

Exercise: Show this!

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$\hat{n} = \hat{a}^\dagger a \rightarrow$ Generator of phase displacements $|\alpha\rangle \rightarrow |\alpha \exp(i\theta)\rangle$

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This lower bound is better by a factor of two than the bound found before, which was $\delta\theta_{\min} = 1/\sqrt{\langle n\rangle}$. This earlier bound corresponds to comparing the displaced-phase coherent state in the upper arm of an interferometer with an undisplaced coherent state with the same amplitude in the other arm.

The result found here indicates that a better measurement of the phase is possible: indeed, a homodyne measurement allows the comparison of the displaced coherent state with a classical reference field (local oscillator), which is just a coherent state with a number of photons much larger than that of the measured state — this yields a better precision in the estimation of the phase.

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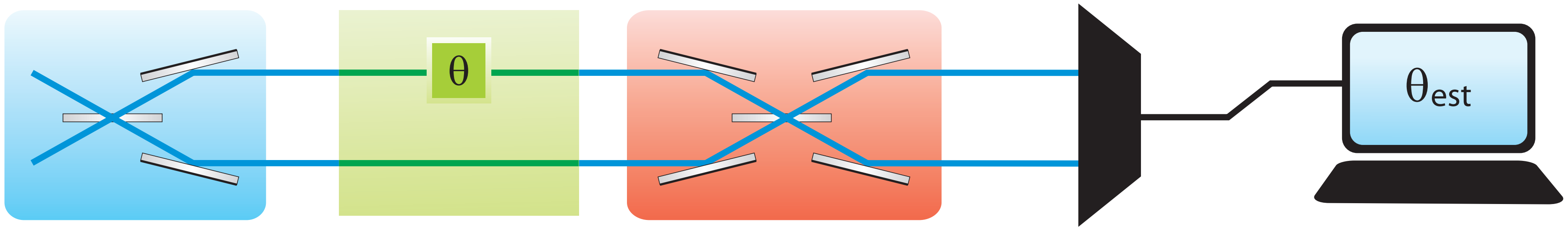

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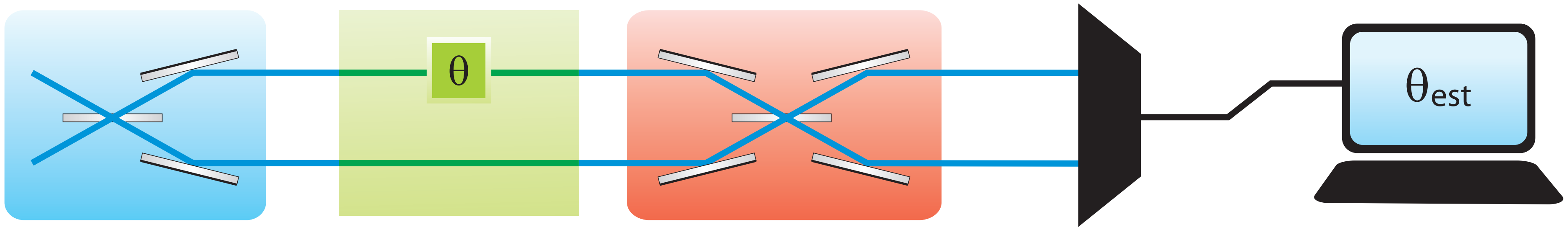

Increasing the precision: maximize variance with NOON states:

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2}$$

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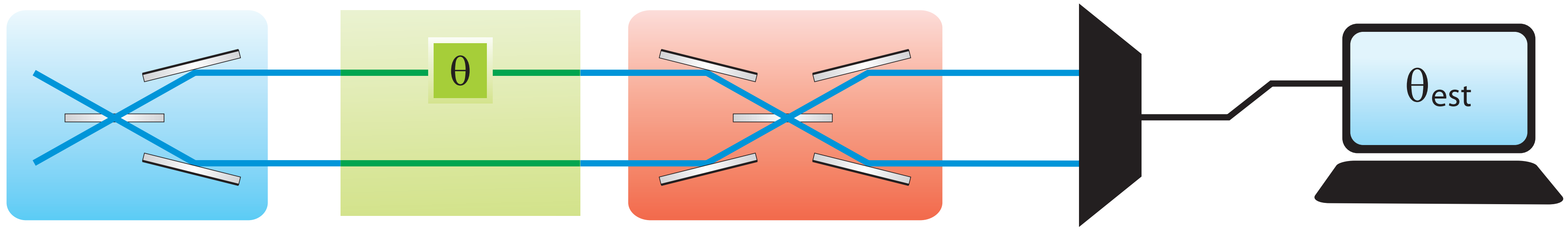

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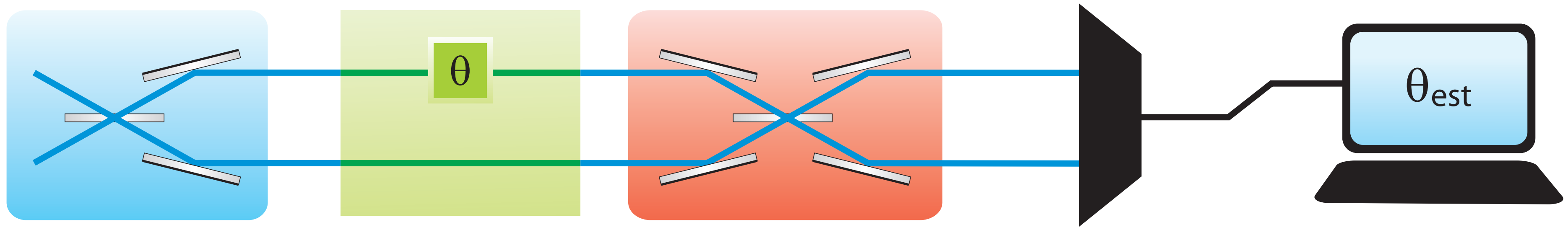

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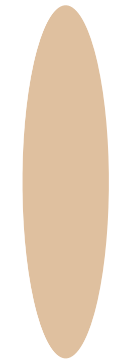
$$\left\langle(\Delta\hat{n})^2\right\rangle_0 = \frac{N^2}{4} \Rightarrow \delta\theta \geq \frac{1}{N}$$

Example 2: Spatial displacement



$$|\psi(X)\rangle = e^{iX\hat{P}}|\psi(0)\rangle \Rightarrow \hat{H} = i\frac{d\hat{U}^\dagger}{dX}\hat{U}(X) = \hat{P}$$

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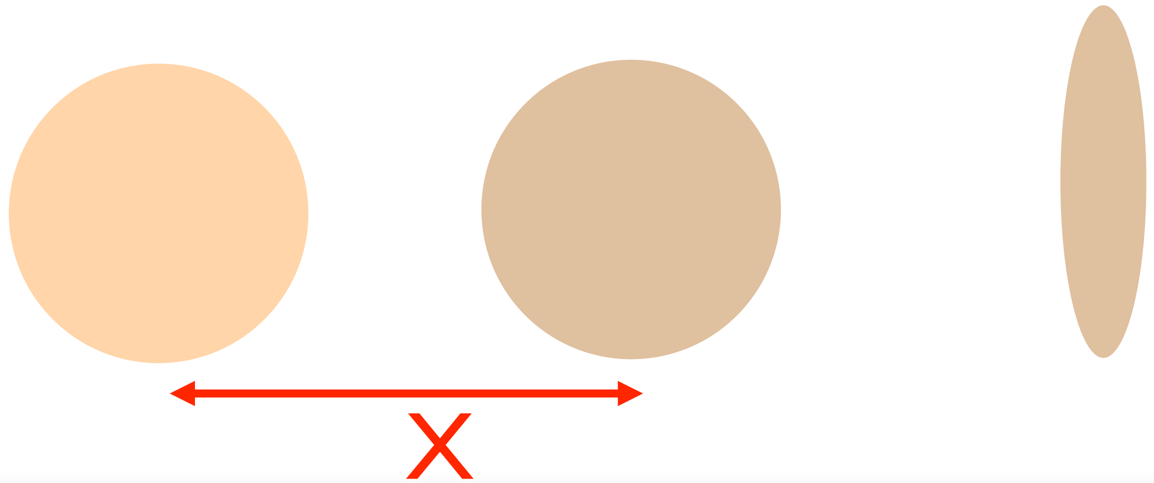

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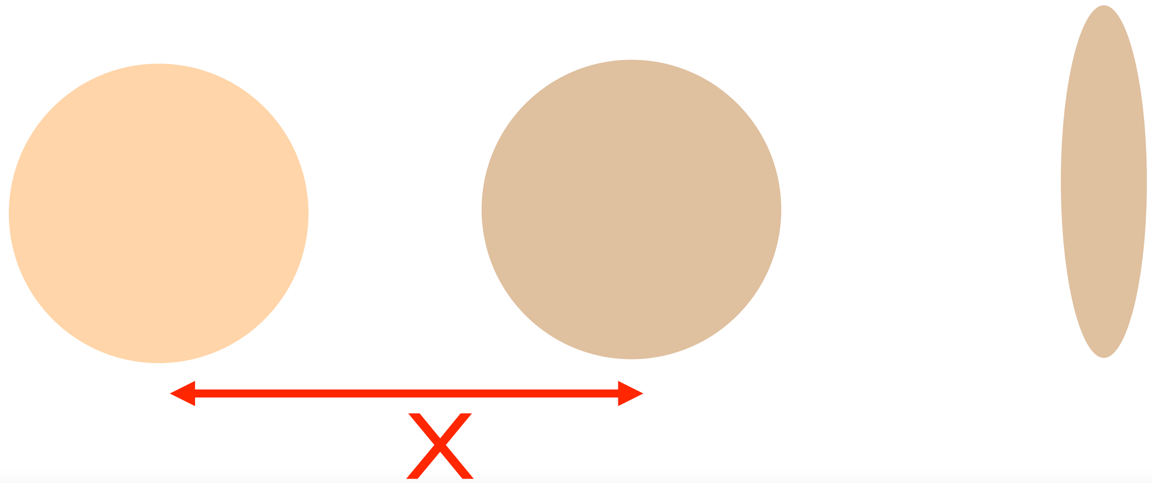

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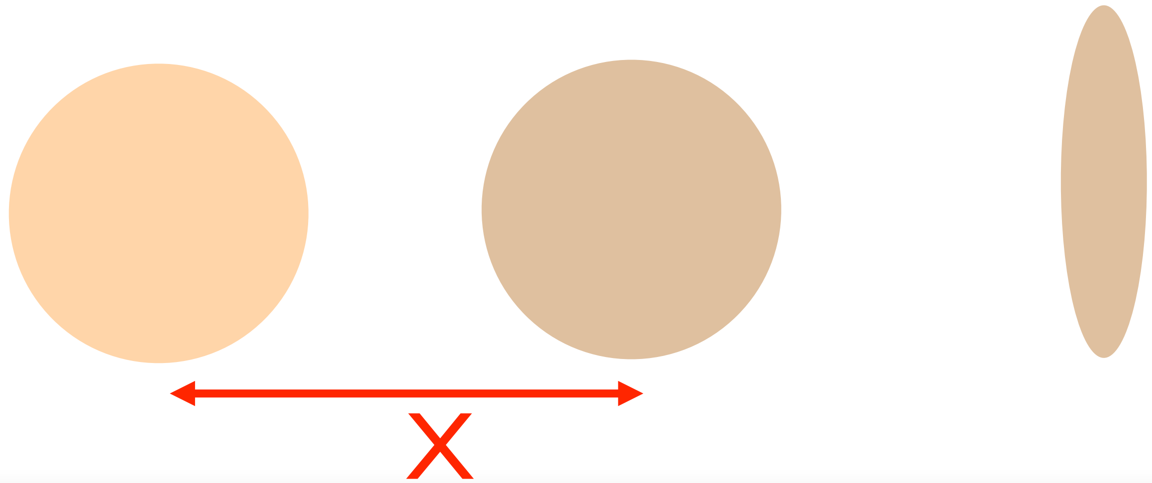

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Maximizing variance of P for better precision: e.g., squeezed states —> Also saturate the bound (Gaussian states)

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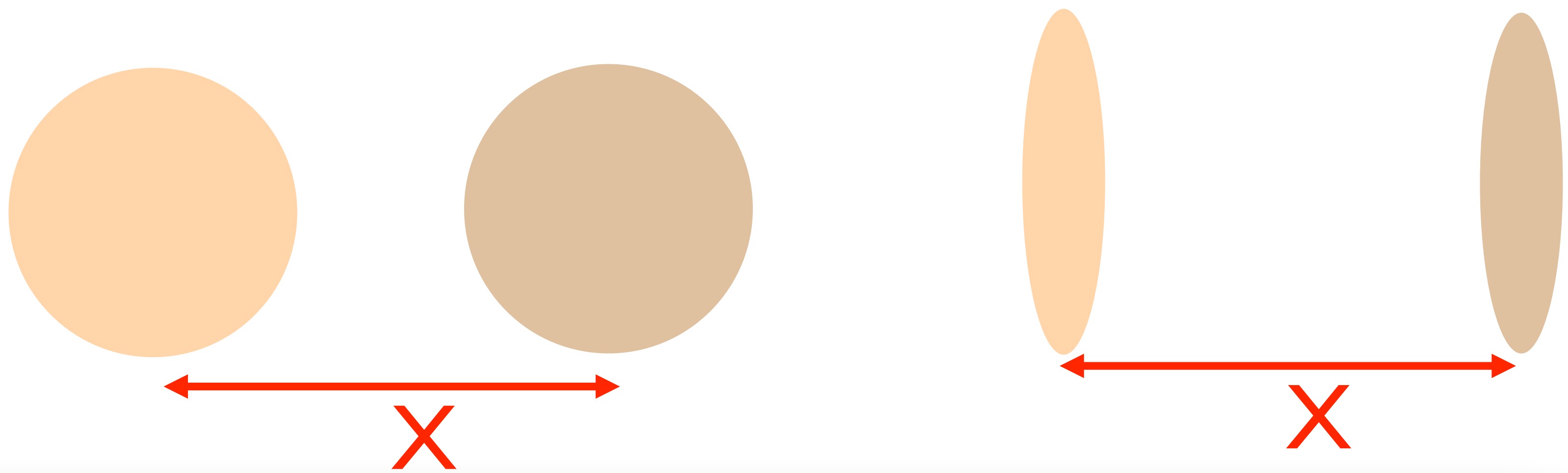


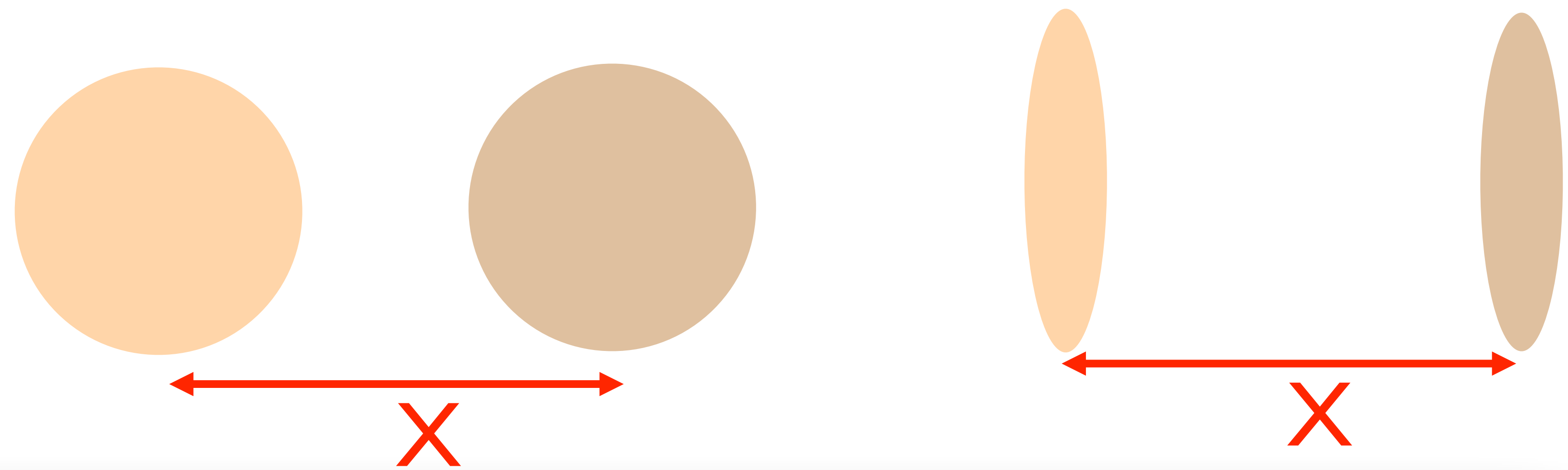
$$|\psi(X)\rangle = e^{iX\hat{P}}|\psi(0)\rangle \Rightarrow \hat{H} = i\frac{d\hat{U}^\dagger}{dX}\hat{U}(X) = \hat{P}$$

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{P})^2\rangle_0 \Rightarrow \langle(\Delta X)^2\rangle \geq \frac{1}{4\langle(\Delta\hat{P})^2\rangle}$$

Coherent state: $\langle(\Delta\hat{P})^2\rangle_0 = 1/2 \Rightarrow \langle(\Delta X)^2\rangle = 1/2$ —> standard quantum limit — coherent state saturates Cramér-Rao bound

Maximizing variance of P for better precision: e.g., squeezed states —> Also saturate the bound (Gaussian states)

Example 2: Spatial displacement



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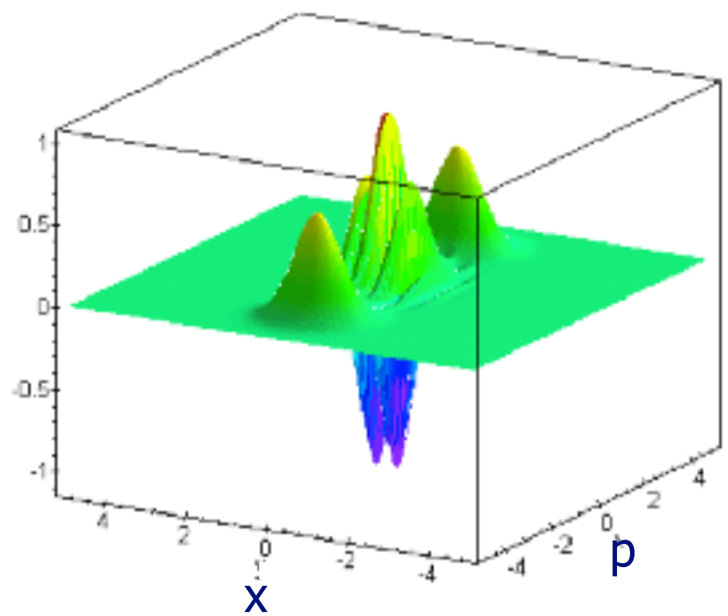
Maximizing variance of P for better precision: e.g., squeezed states —> Also saturate the bound (Gaussian states)

Looks like Heisenberg uncertainty relation, but X is a parameter, not an operator!

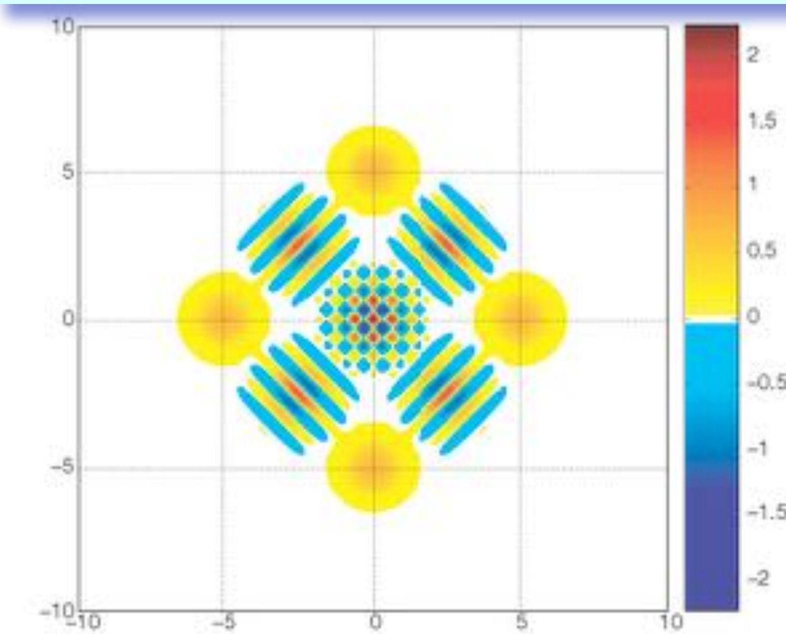
Example 3: Phase-space displacement

A sensitive instrument...



$$|\psi\rangle = \mathrm{N}\left(|\alpha\rangle + |-\alpha\rangle\right)$$

W. Zurek, Nature **412**, 712 (2001)



Vlastakis et al., Science **342**, 607 (2013)

$$|\psi\rangle = \mathrm{N}'\left(|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle\right)$$

$$\Delta X \approx \frac{1}{|\alpha|}$$

Sub-Planck sensitivity

PHYSICAL REVIEW A 73, 023803 (2006)

Sub-Planck phase-space structures and Heisenberg-limited measurements

F. Toscano,[1] D. A. R. Dalvit,[2] L. Davidovich,[1] and W. H. Zurek[2]

PHYSICAL REVIEW A 94, 022313 (2016)

Measurement of a microwave field amplitude beyond the standard quantum limit

M. Penasa,[1] S. Gerlich,[1] T. Rybarczyk,[1] V. Métillon,[1] M. Brune,[1] J. M. Raimond,[1] S. Haroche,[1] L. Davidovich,[2] and I. Dotsenko[1,*]

Possible strategies for quantum-enhanced metrology (1)

Single probe

Recall that $\mathcal{F}_Q(|\psi\rangle) = 4\langle(\Delta\hat{H})^2\rangle$ so in order to increase the precision one needs to choose a state $|\psi\rangle$ that maximizes the variance $\langle(\Delta\hat{H})^2\rangle$. If $\hat{H}$ has a discrete and bounded spectrum, this is accomplished by letting

$$|\psi\rangle_{\rm opt} = \frac{1}{\sqrt{2}}\left(|\lambda_{\rm max}\rangle + |\lambda_{\rm min}\rangle\right)$$

where $|\lambda_{\rm max}\rangle$ and $|\lambda_{\rm min}\rangle$ are eigenstates of $\hat{H}$ corresponding to the maximum and minimum eigenvalues.

Then $\langle(\Delta\hat{H})^2\rangle = (\lambda_{\rm max} - \lambda_{\rm min})^2/4$ and

$$\Delta\varphi_{(1)} \geq \frac{1}{\sqrt{\nu}\left(\lambda_{\rm max} - \lambda_{\rm min}\right)}$$

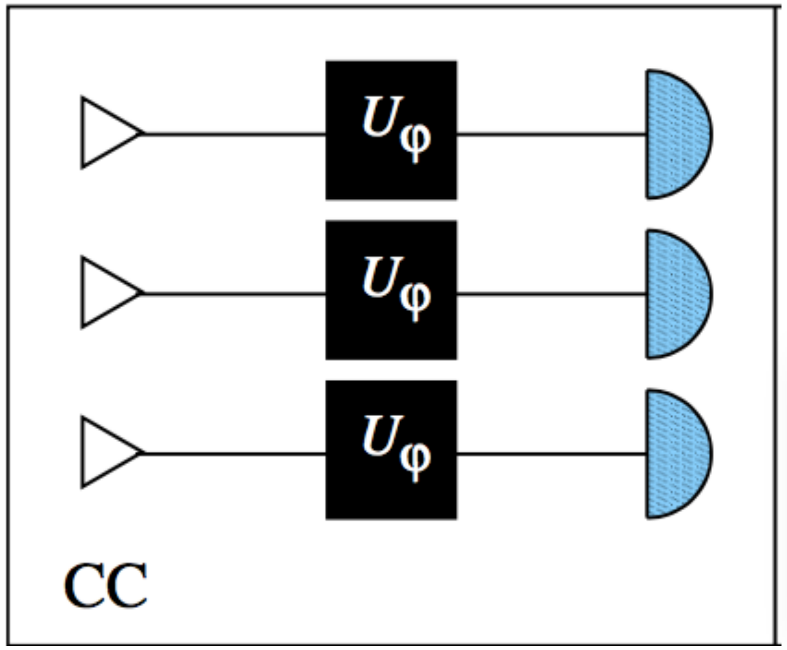
(ν —> number of repetitions of single probe experiment)

Question: What is the best strategy if one has N probes?

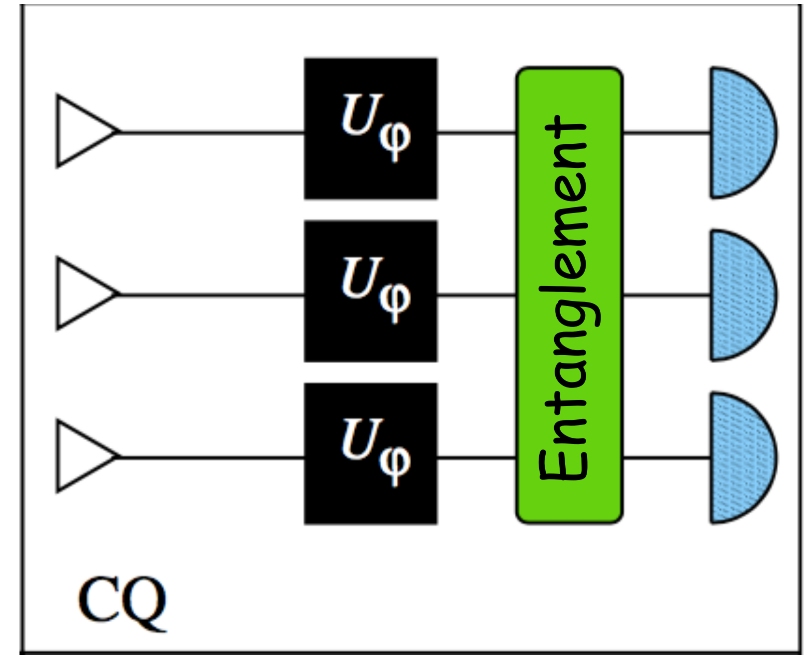
Possible strategies for quantum-enhanced metrology (2)

N probes

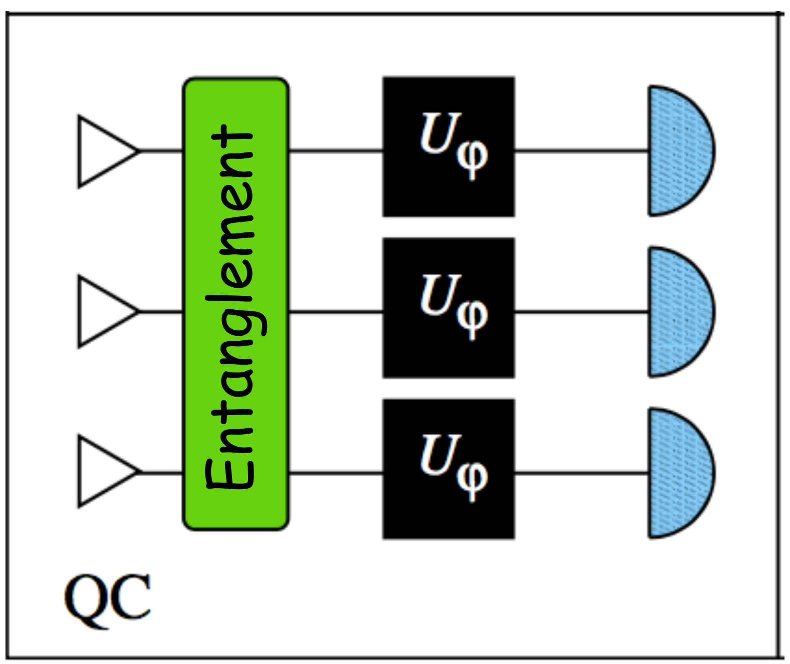
V. Giovannetti, S. Lloyd, and L. Macone, PRL **96**, 010401 (2006)



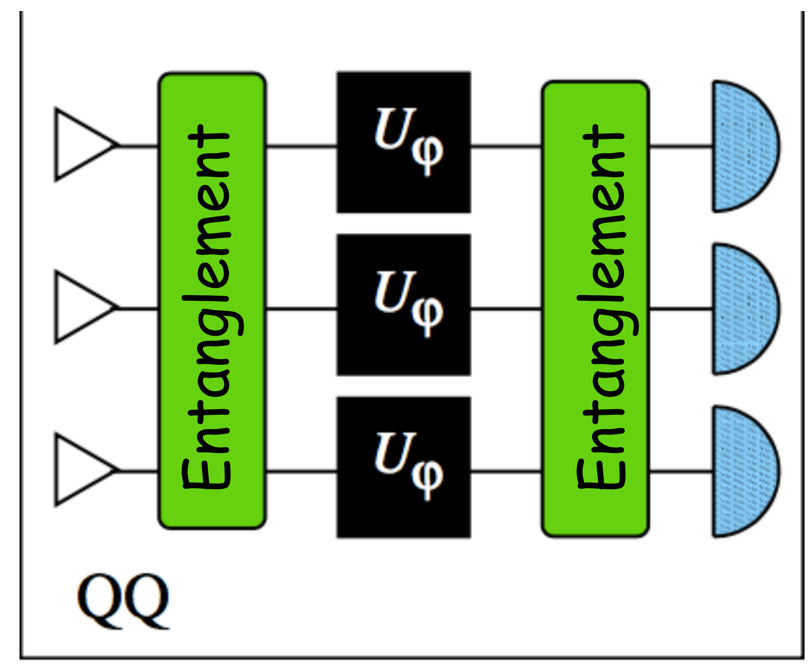
Separable input states, separable measurements



Separable input states, general measurement schemes (with entanglement)

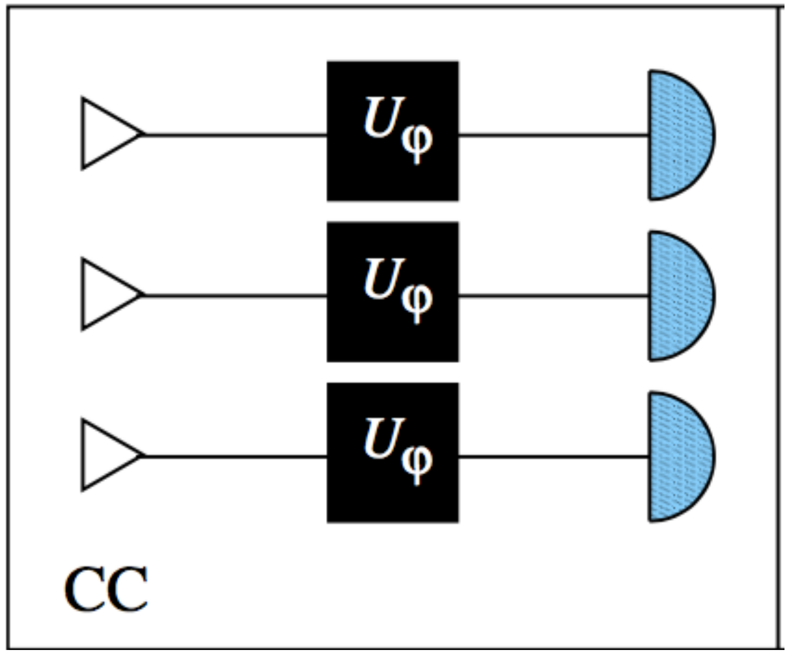


General input states (with entanglement), separable measurements

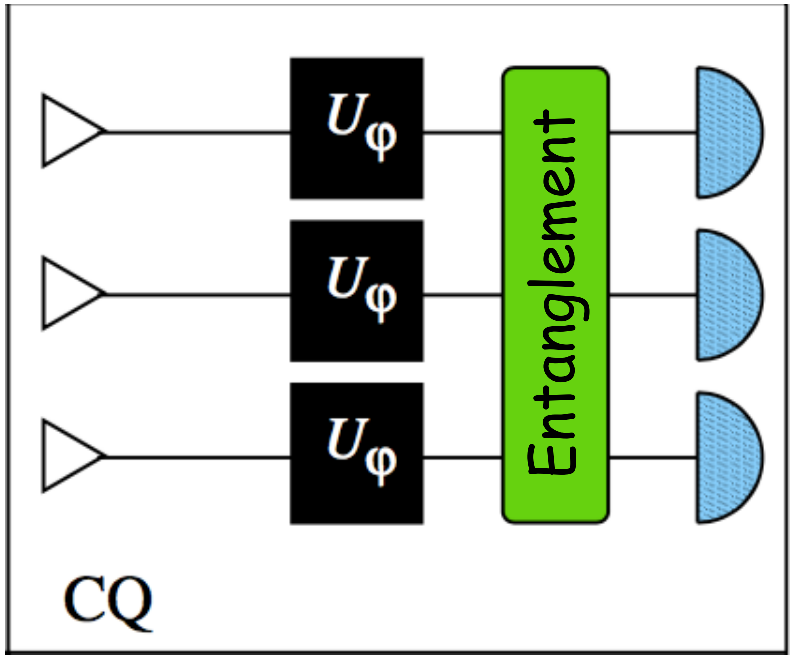


General input states, general measurement schemes (with entanglement)

Possible strategies for quantum-enhanced metrology (3)



N probes



Separable input states, separable measurements

Separable input states, general measurement schemes (including entanglement)

$$\hat{U}^{(N)}(\varphi) = \hat{U}(\varphi)^{\otimes N} \qquad \hat{\mathcal{H}} = \sum_{j=1}^{N} \hat{H}_j \rightarrow \text{ generators of } \hat{U}(\varphi)$$

Product initial state: $\left\langle \Delta \hat{\mathcal{H}}^2 \right\rangle = \sum_{j=1}^{N} \left\langle \Delta \hat{H}_j^2 \right\rangle_{|\psi_j\rangle}$

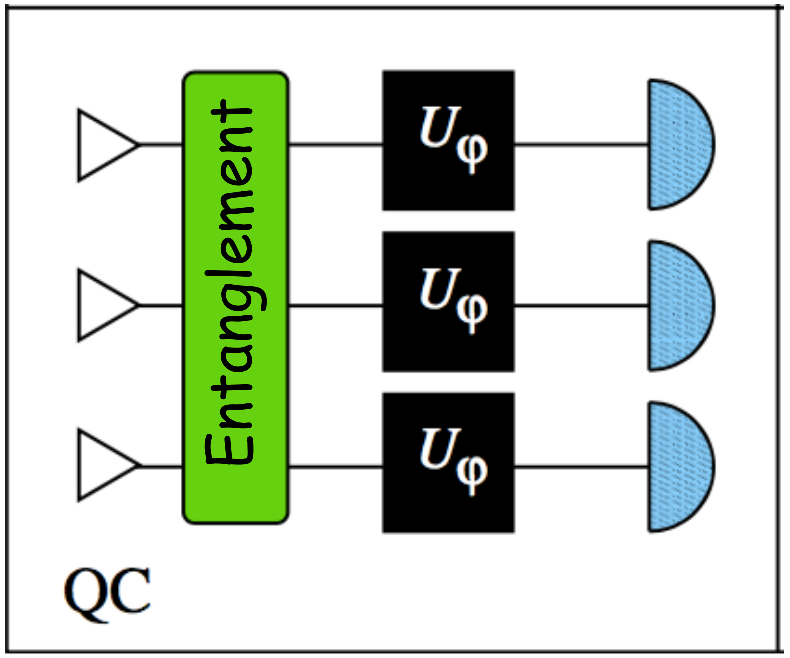
$$|\Psi\rangle_{\text{opt}} = |\psi\rangle_{\text{opt}}^{(1)} \otimes |\psi\rangle_{\text{opt}}^{(2)} \otimes \cdots \otimes |\psi\rangle_{\text{opt}}^{(N)} \rightarrow \left\langle \Delta \hat{\mathcal{H}}^2 \right\rangle = N(\lambda_{\max} - \lambda_{\min})^2/4$$

Therefore

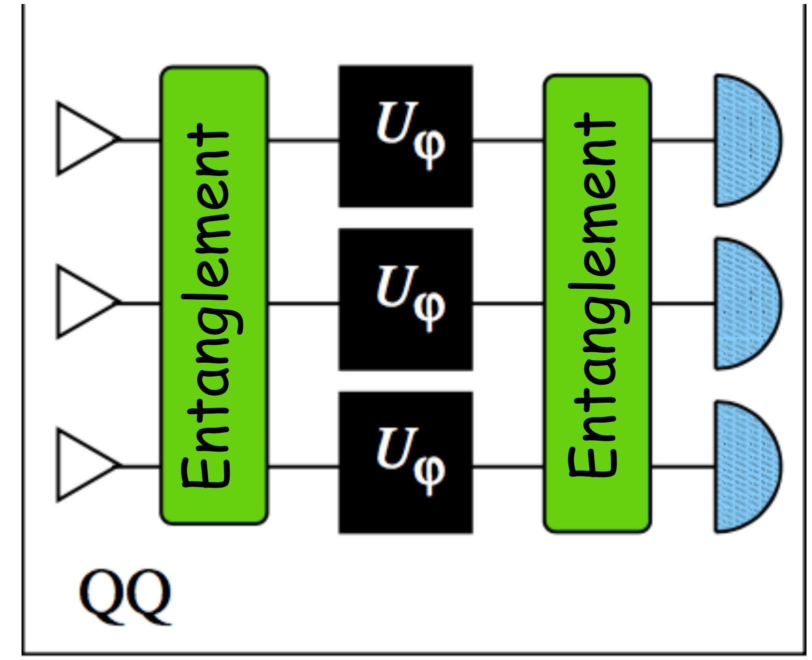
$$\Delta\varphi_{(N)} \geq \frac{1}{\sqrt{\nu N}\,(\lambda_{\max} - \lambda_{\min})} = \frac{\Delta\varphi_{(1)}}{\sqrt{N}}$$

Possible strategies for quantum-enhanced metrology (4)

N probes



General input states, separable measurements



General input states, general measurement schemes

$$\hat{U}^{(N)}(\varphi) = \hat{U}(\varphi)^{\otimes N} \qquad \hat{\mathcal{H}} = \sum_{j=1}^{N} \hat{H}_j$$

Maximization of variance $\langle (\Delta\hat{\mathcal{H}})^2 \rangle$:

$$|\Psi\rangle_{\text{opt}} = \frac{1}{\sqrt{2}}\left(|\lambda_{\max}\rangle_1 \otimes |\lambda_{\max}\rangle_2 \otimes \cdots \otimes |\lambda_{\max}\rangle_N + |\lambda_{\min}\rangle_1 \otimes |\lambda_{\min}\rangle_2 \otimes \cdots \otimes |\lambda_{\min}\rangle_N\right)$$

$$\langle (\Delta\hat{\mathcal{H}})^2 \rangle = N^2(\lambda_{\max} - \lambda_{\min})^2/4$$

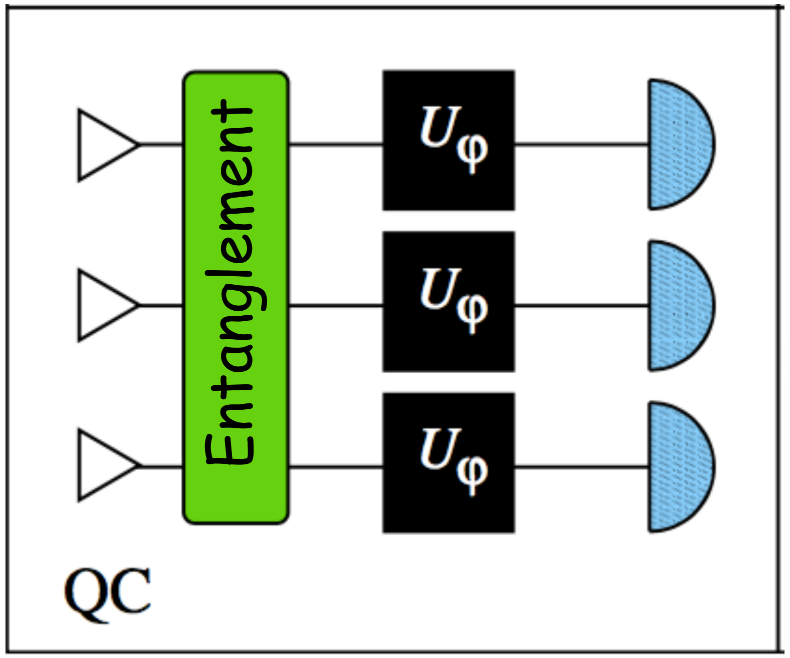
Therefore: $$\Delta\varphi_{(N)} \geq \frac{1}{N\sqrt{\nu}\,(\lambda_{\max} - \lambda_{\min})} = \frac{\Delta\varphi_{(1)}}{N}$$

$1/\sqrt{N}$ gain!

—> Heisenberg limit

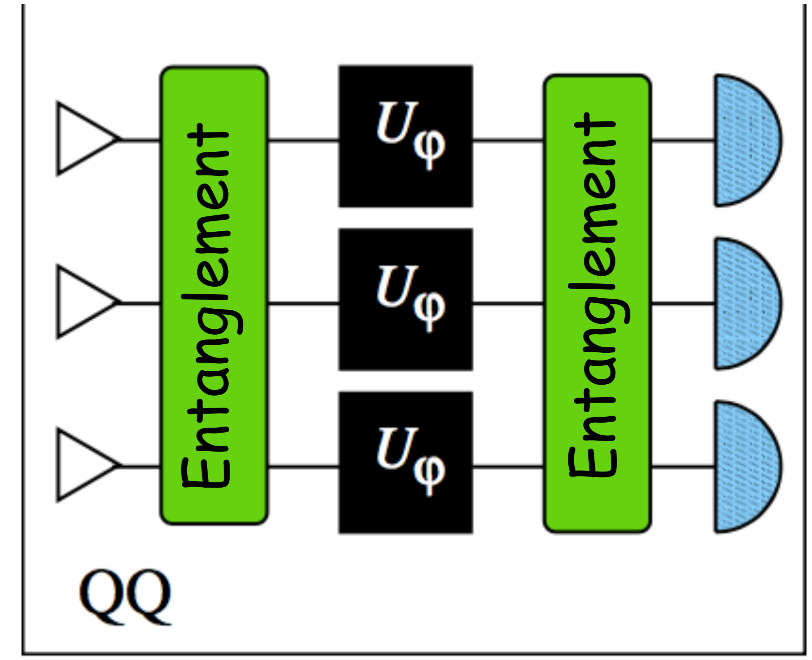
Possible strategies for quantum-enhanced metrology (4)

N probes



General input states, separable measurements

Entanglement of initial state is necessary for going beyond shot-noise scaling.



General input states, general measurement schemes

$$\hat{U}^{(N)}(\varphi) = \hat{U}(\varphi)^{\otimes N} \qquad \hat{\mathcal{H}} = \sum_{j=1}^{N} \hat{H}_j$$

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$$|\Psi\rangle_{\rm opt} = \frac{1}{\sqrt{2}}\left(|\lambda_{\max}\rangle_1 \otimes |\lambda_{\max}\rangle_2 \otimes \cdots \otimes |\lambda_{\max}\rangle_N + |\lambda_{\min}\rangle_1 \otimes |\lambda_{\min}\rangle_2 \otimes \cdots \otimes |\lambda_{\min}\rangle_N\right)$$

$$\langle (\Delta\hat{\mathcal{H}})^2 \rangle = N^2(\lambda_{\max} - \lambda_{\min})^2/4$$

Therefore: $$\Delta\varphi_{(N)} \geq \frac{1}{N\sqrt{\nu}\,(\lambda_{\max} - \lambda_{\min})} = \frac{\Delta\varphi_{(1)}}{N}$$

$1/\sqrt{N}$ gain!

—> Heisenberg limit

Entanglement-assisted parameter estimation: phase estimation

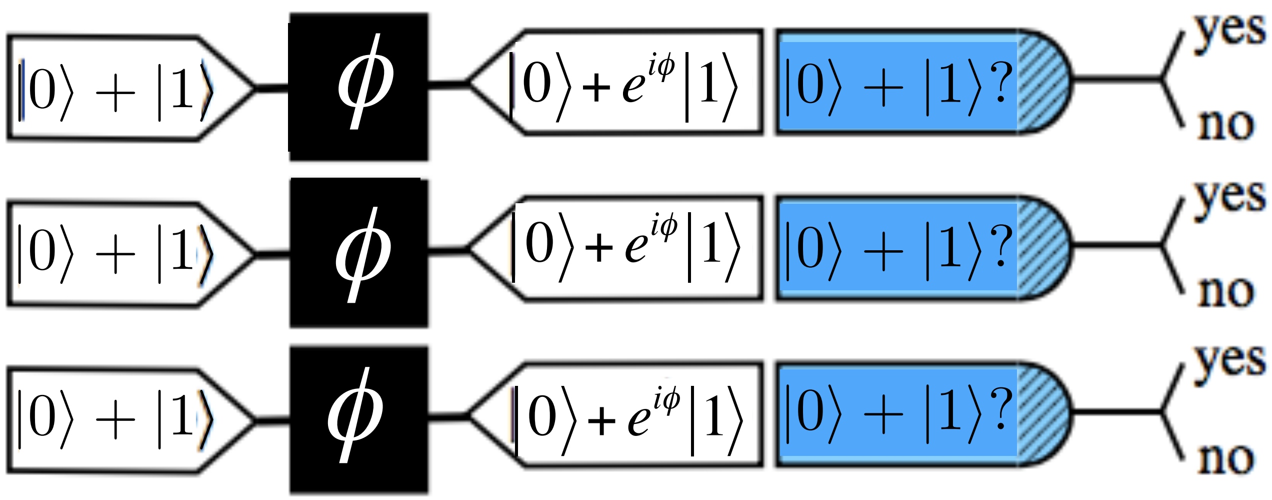
The problem. One wants to estimate a small change of phase between states of a two-level system, which would allow to estimate say a small electromagnetic field, or yet a transition frequency between the two states.

Two possible strategies:

[Figures adapted from V. Giovannetti, S. Lloyd and L. Maccone, Nature Photonics **5**, 222–229 (2011)]

Separable

$$\left(|0\rangle+|1\rangle\right)\rightarrow \exp\left[i\left(1+\hat{\sigma}_z\right)\phi/2\right]\left(|0\rangle+|1\rangle\right)$$



$$p_S(\text{yes})\equiv p_S=\left(1+\cos\phi\right)/2$$

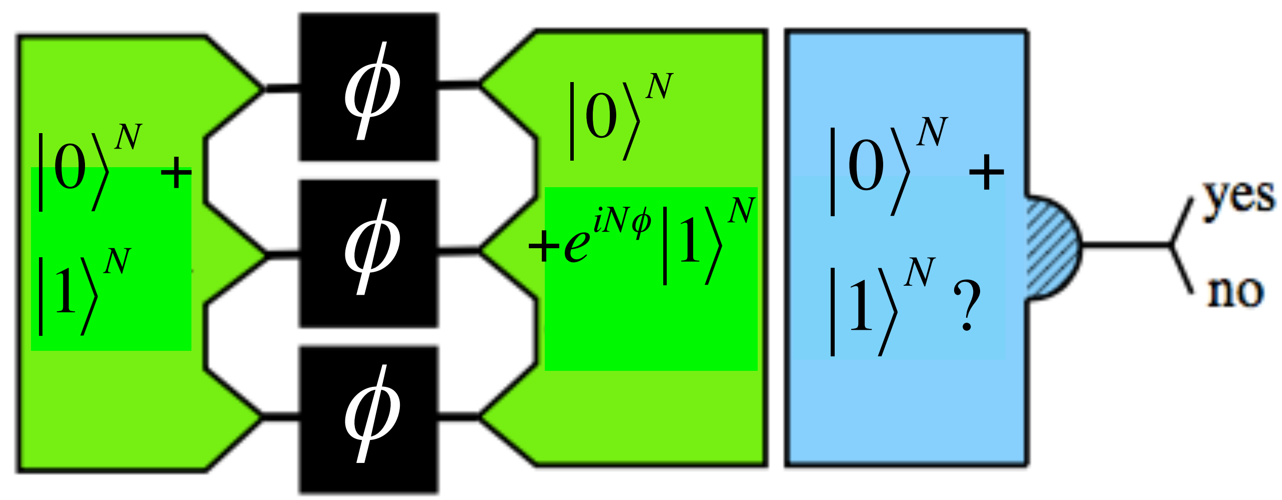
$$p_S(\text{no})=1-p_S=\left(1-\cos\phi\right)/2$$

$$F_S(\phi)=\left(\frac{1}{p_S}+\frac{1}{1-p_S}\right)\left[\frac{\partial p_s}{\partial\phi}\right]^2=\left[\frac{1}{p_S\left(1-p_S\right)}\right]\left[\frac{\partial p_s}{\partial\phi}\right]^2$$

$$=1$$

$$\delta\phi_S\geq 1/\sqrt{NF_S(\phi)}=1/\sqrt{N}$$

Entangled



$$p_E(\text{yes})\equiv p_E=\left(1+\cos N\phi\right)/2$$

$$p_E(\text{no})=1-p_E=\left(1-\cos N\phi\right)/2$$

$$F_E(\phi)=\left[\frac{1}{p_E\left(1-p_E\right)}\right]\left[\frac{\partial p_E}{\partial\phi}\right]^2=1$$

$$\delta\phi_E\geq 1/\sqrt{NF_E(\phi)}=1/N$$

Entanglement-assisted parameter estimation: phase estimation (2)

Are these the best measurements?

$$\left(|0\rangle+|1\rangle\right)\rightarrow \exp\left[i\left(1+\hat{\sigma}_z\right)\phi/2\right]\left(|0\rangle+|1\rangle\right)$$

1. Separable qubits.

Entanglement-assisted parameter estimation: phase estimation (2)

Are these the best measurements?

$$\left(|0\rangle + |1\rangle\right) \to \exp\left[i\left(1+\hat{\sigma}_z\right)\phi/2\right]\left(|0\rangle + |1\rangle\right)$$

1. Separable qubits.

We know that for the best measurement $\mathcal{F}_Q(\phi) = 4\langle(\Delta\hat{H})^2\rangle_0$, where $\hat{H}$ here is the generator of phase displacements: $\hat{H} = (1 + \hat{\sigma}_z)/2$. Since for the initial state $|+\rangle$ we have $\langle(\Delta\hat{H})^2\rangle_0 = 1/4$, it follows that the measurement of $\hat{\sigma}_x$ maximizes the Fisher information, leading to the corresponding Cramér-Rao bound in $\delta\phi \geq 1/\sqrt{N\mathcal{F}_Q(\phi)} = 1/\sqrt{N}$, the so-called standard limit.

Entanglement-assisted parameter estimation: phase estimation (2)

Are these the best measurements?

$$\big(|0\rangle+|1\rangle\big)\rightarrow \exp\big[i(1+\hat{\sigma}_z)\phi/2\big]\big(|0\rangle+|1\rangle\big)$$

1. Separable qubits.

We know that for the best measurement $\mathcal{F}_Q(\phi)=4\langle(\Delta\hat{H})^2\rangle_0$, where $\hat{H}$ here is the generator of phase displacements: $\hat{H}=(1+\hat{\sigma}_z)/2$. Since for the initial state $|+\rangle$ we have $\langle(\Delta\hat{H})^2\rangle_0=1/4$, it follows that the measurement of $\hat{\sigma}_x$ maximizes the Fisher information, leading to the corresponding Cramér-Rao bound in $\delta\phi\geq 1/\sqrt{N\mathcal{F}_Q(\phi)}=1/\sqrt{N}$, the so-called standard limit.

2. Entangled qubits.

The generator of phase displacements is $\hat{H}=\sum_{i=1}^{N}\left(1+\hat{\sigma}_z^{(i)}\right)/2$, so that $\langle\psi(0)|(\Delta\hat{H})^2|\psi(0)\rangle=N^2/4$, which means that the above measurement leads to the maximum value of the Fisher information and to the Cramér-Rao bound in $\delta\phi\geq 1/\sqrt{\mathcal{F}_Q(\phi)}=1/N$, the Heisenberg limit.

Entanglement-assisted parameter estimation: phase estimation (3)

2. Entangled qubits.

Bound can be achieved with local measurements! Measure observable

$\hat{\sigma}^{\otimes N} =$ on final state $|0\rangle^N + e^{iN\phi}|1\rangle^N$

Get

$$\langle \hat{\sigma}^{\otimes N} \rangle = \cos(N\varphi)$$

$$\Delta\hat{\sigma}^{\otimes N} = |\sin(N\varphi)|$$

So, from error propagation:

$$\delta\varphi = \frac{\Delta\hat{\sigma}^{\otimes N}}{\partial\langle\hat{\sigma}^{\otimes N}\rangle / \partial\varphi} = \frac{1}{N}$$

which coincides with the Heisenberg bound.

Therefore, only the initial entanglement counts!

Entanglement-assisted parameter estimation: phase estimation (3)

2. Entangled qubits.

Bound can be achieved with local measurements! Measure observable $\hat{\sigma}^{\otimes N} = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} \cdots \otimes \hat{\sigma}_x^{(N)}$ on final state $|0\rangle^N + e^{iN\phi}|1\rangle^N$

Get

$$\langle \hat{\sigma}^{\otimes N} \rangle = \cos(N\varphi)$$

$$\Delta \hat{\sigma}^{\otimes N} = |\sin(N\varphi)|$$

So, from error propagation:

$$\delta\varphi = \frac{\Delta \hat{\sigma}^{\otimes N}}{\partial \langle \hat{\sigma}^{\otimes N} \rangle / \partial \varphi} = \frac{1}{N}$$

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NEXT LECTURE: QUANTUM METROLOGY FOR OPEN SYSTEMS