Quantum physics frontiers
explored with cold atoms, molecules and photons
Heraklion Crete, July 24-28, 2017

Instituto de Física UFRJ

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

Luiz Davidovich
Instituto de Física - Universidade Federal do Rio de Janeiro

LECTURE 2

THE ONASSIS FOUNDATION
SCIENCE LECTURE SERIES

Classical parameter estimation

H. Cramér

C. R. Rao

R.A. Fisher

Cramér-Rao bound for unbiased estimators:

$$\Delta X \geq 1/\sqrt{N\,F(X)\big|_{X=X_{\text{true}}}}\,,\quad F(X) \equiv \sum_j P_j(X)\left(\frac{d\,\ln\left[P_j(X)\right]}{dX}\right)^2$$

$N \rightarrow$ Number of repetitions of the experiment

$P_j(X) \rightarrow$ probability of getting an experimental result j

or yet, for continuous measurements: $F(X) \equiv \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2$

where ξ are the measurement results

(Average over all experimental results)

Quantum Fisher Information

(Helstrom, Holevo, Braunstein and Caves)

$$F\left(X;\{\hat{E}_\xi\}\right) \equiv \int d\xi \, p(\xi \mid X)\left(\frac{d \ln\left[p(\xi \mid X)\right]}{dX}\right)^2$$

$$p(\xi \mid X) = \mathrm{Tr}\left[\hat{\rho}(X)\hat{E}_\xi\right]$$

$$\int d\xi \hat{E}_\xi = \hat{1}$$

POVM

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This corresponds to a given quantum measurement. Ultimate lower bound for $\langle(\Delta X_{\mathrm{est}})^2\rangle$: optimize over all quantum measurements so that

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Initial state of the probe: $|\psi(0)\rangle$
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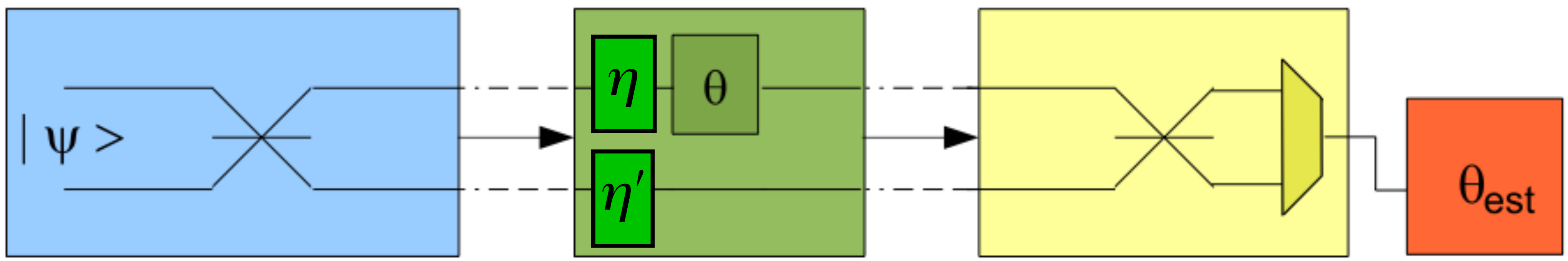
Parameter estimation with decoherence



Loss of a single photon transforms NOON state into a separable state!

$$|\psi(N)\rangle = \frac{|N,0\rangle + |0,N\rangle}{\sqrt{2}} \rightarrow |N-1,0\rangle \text{ or } |0,N-1\rangle$$

No simple analytical expression for Fisher information!

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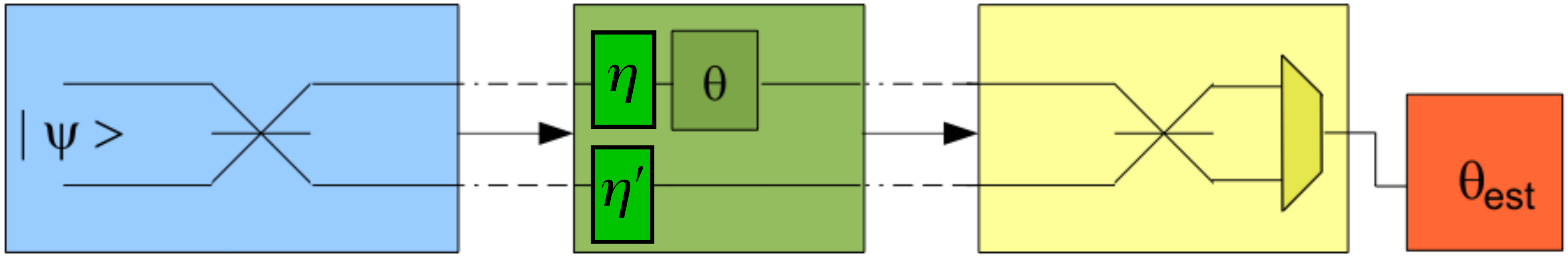

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Experimental test with more robust states (for N=2):

nature photonics

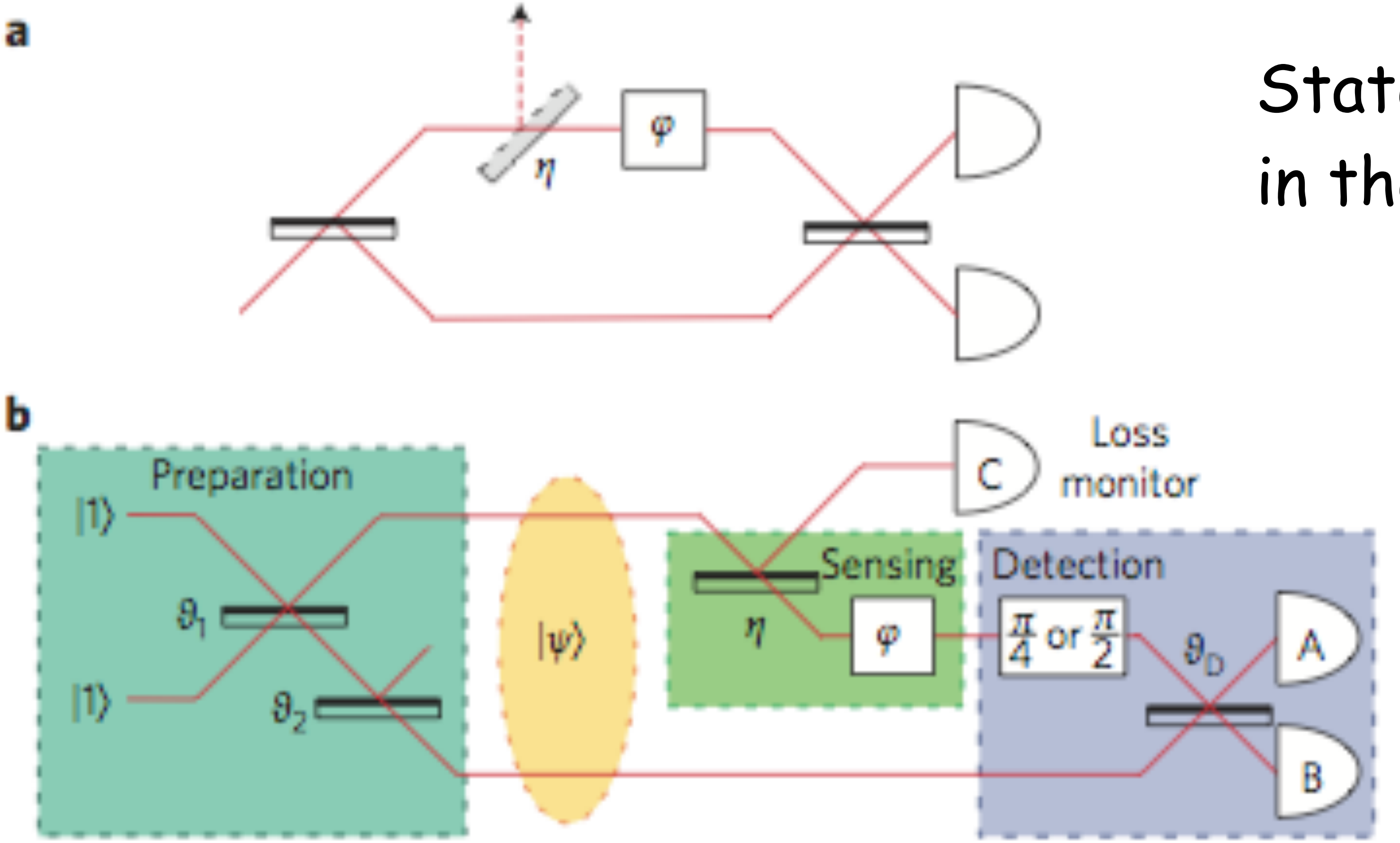
LETTERS

PUBLISHED ONLINE: 4 APRIL 2010 | DOI: 10.1038/NPHOTON.2010.39

Experimental quantum-enhanced estimation of a lossy phase shift

M. Kacprowicz[1], R. Demkowicz-Dobrzański[1,2]*, W. Wasilewski[2], K. Banaszek[1,2] and I. A. Walmsley[3]

Parameter estimation with losses - experiments



States leading to minimum uncertainty in the presence of noise:

$$|\psi\rangle = \sqrt{x_2}\,|20\rangle + \sqrt{x_1}\,|11\rangle - \sqrt{x_0}\,|02\rangle$$

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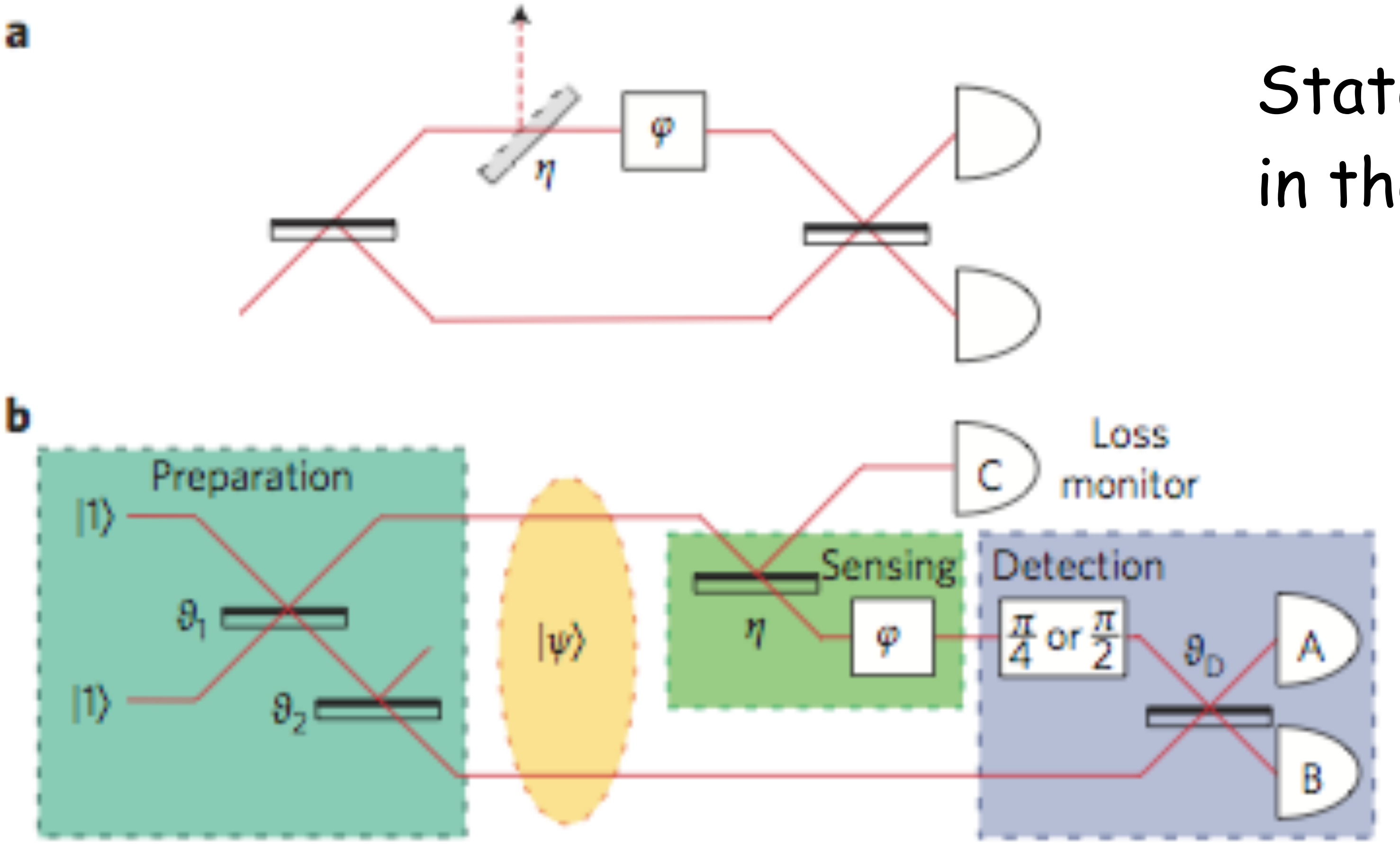

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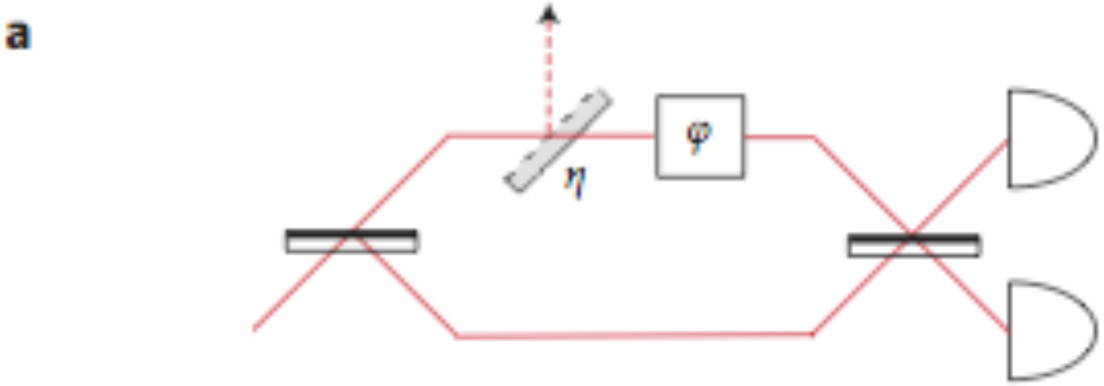

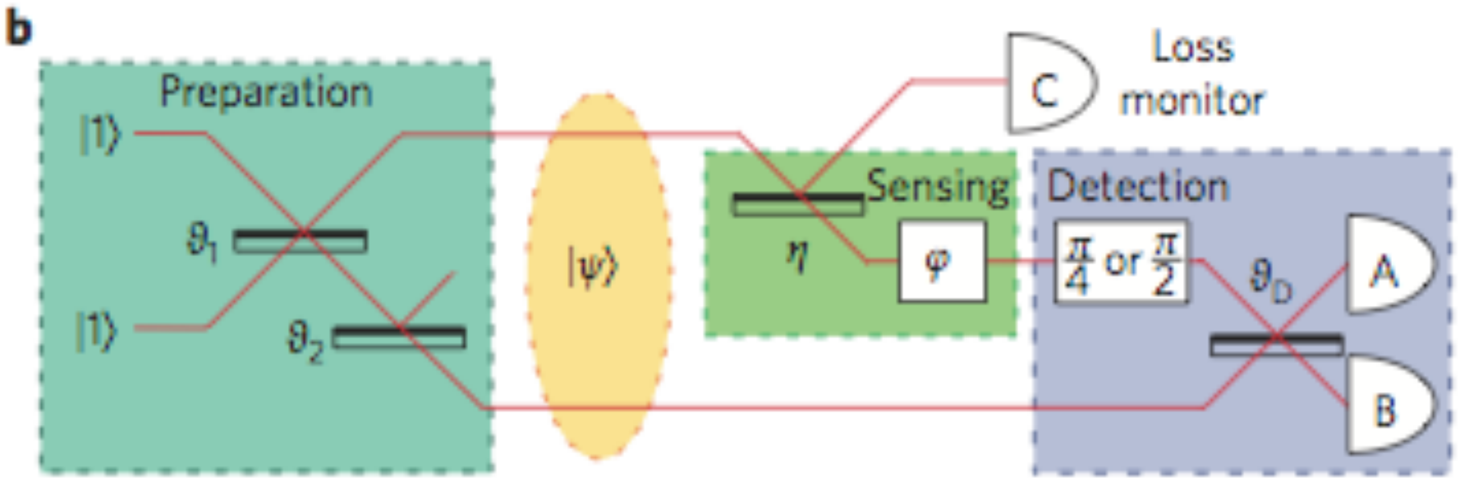

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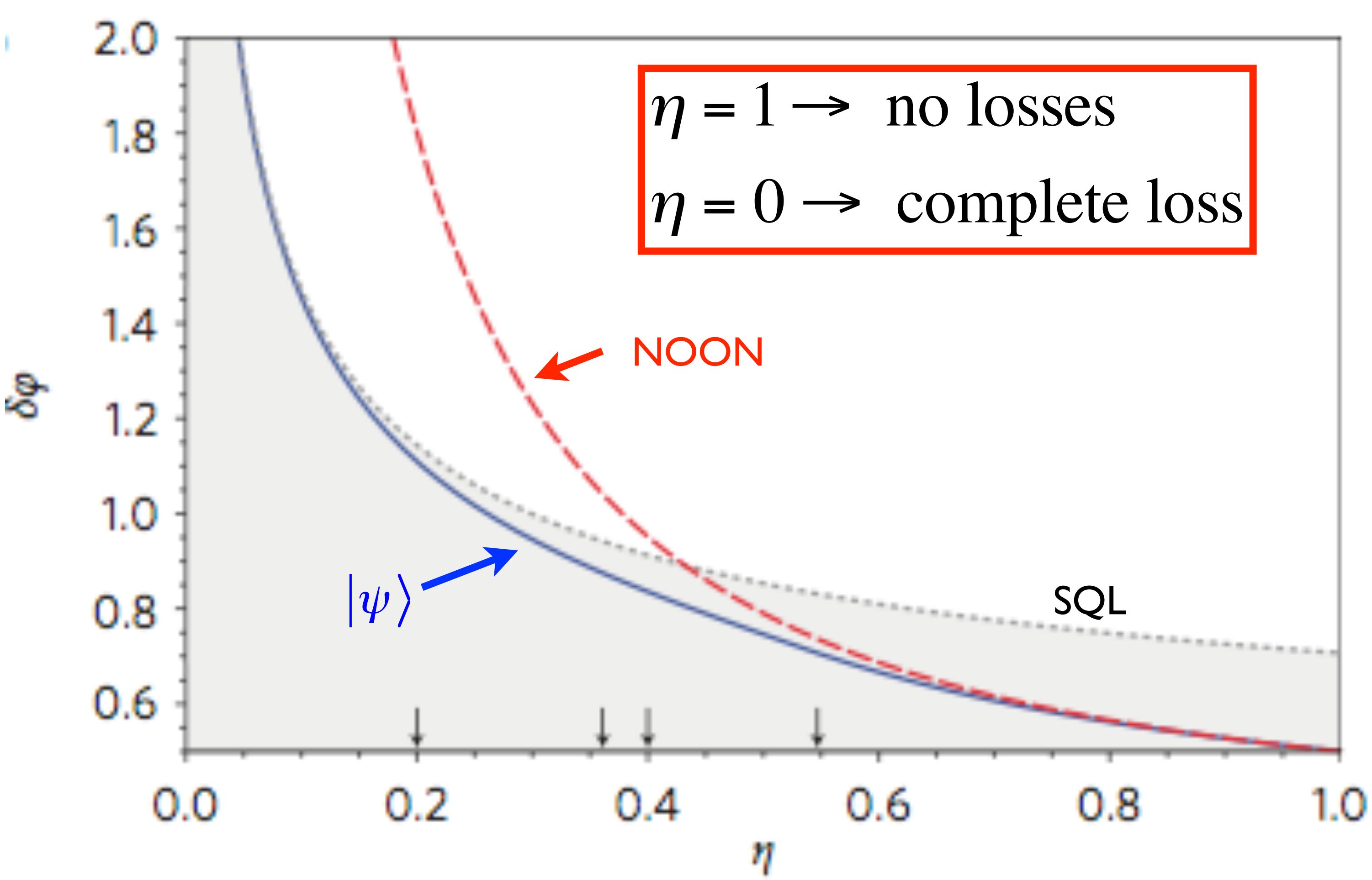

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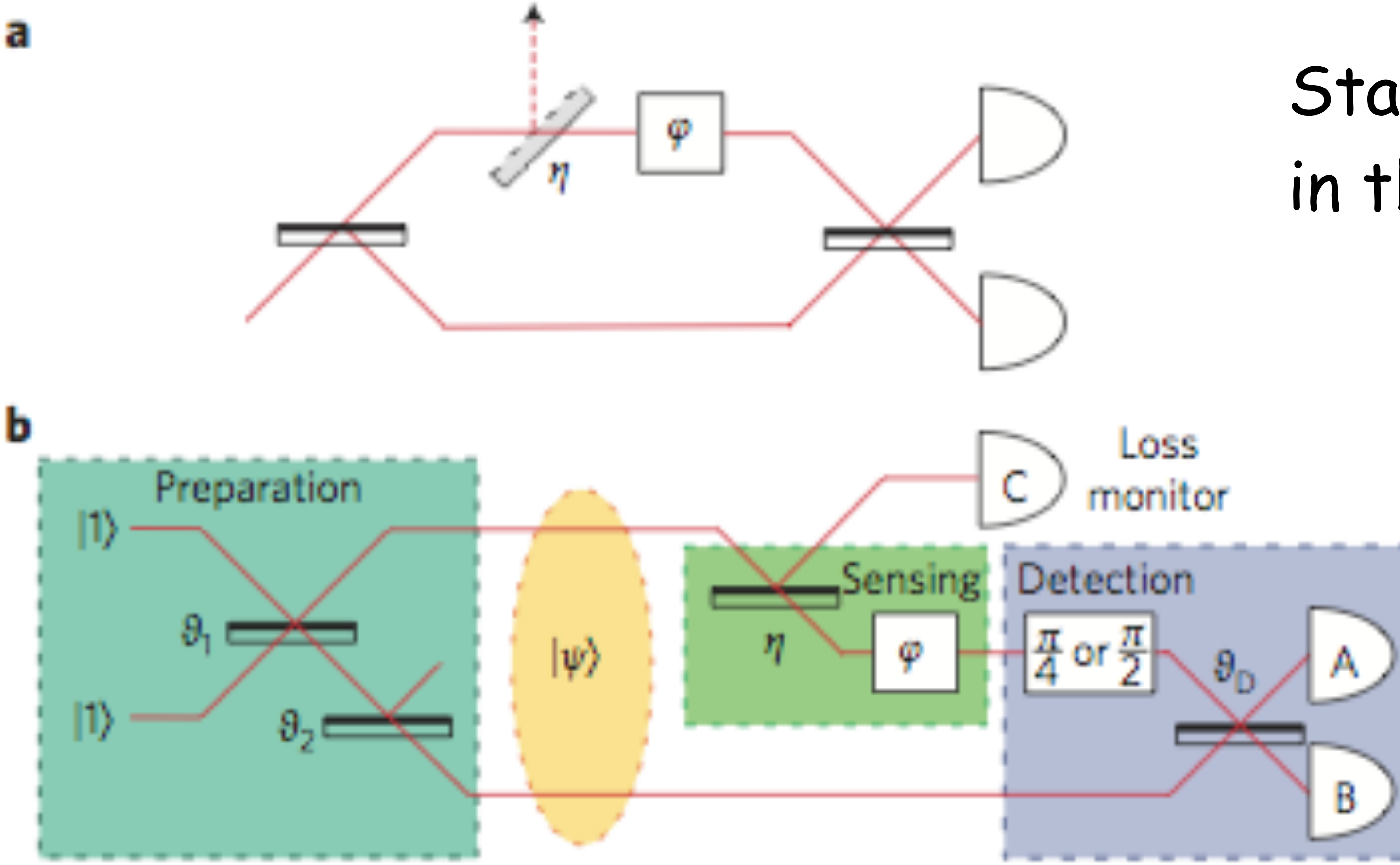

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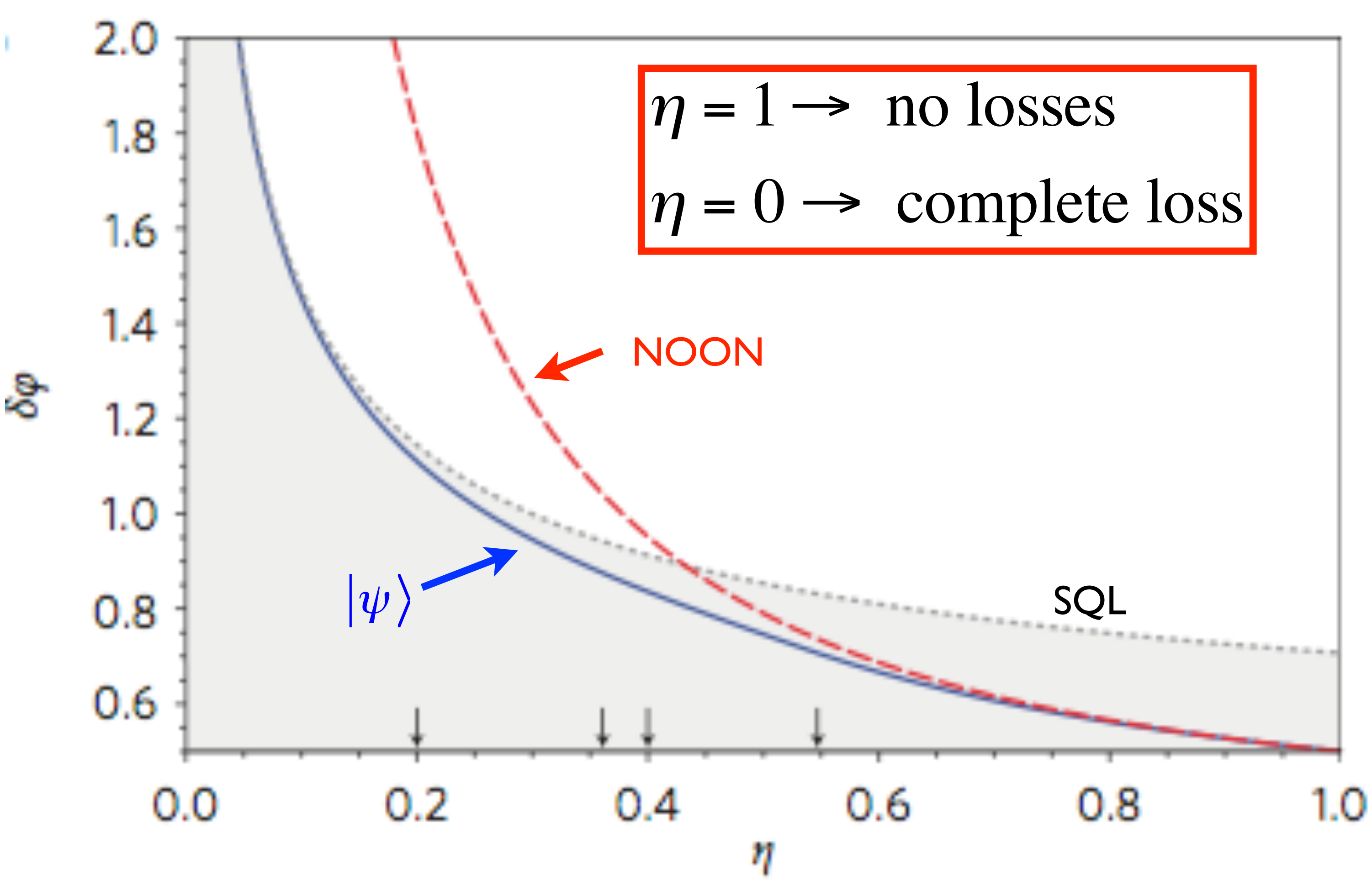

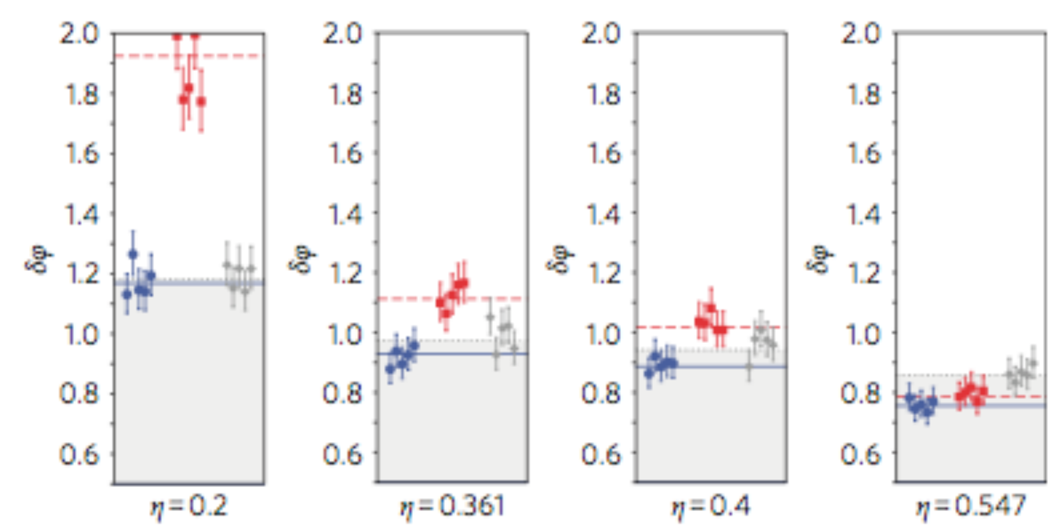

Figure 5 | Uncertainty of phase estimates. Uncertainties obtained using two-photon optimal (circles) and NOON (squares) states, as well as attenuated laser pulses in the SIL regime (diamonds), rescaled by the square root of the number of coincidences. For each transmission η, data are shown for five phases $\varphi = 0, \pm 0.2, \pm 0.4$ rad. Horizontal lines represent the theoretical Cramér-Rao bounds for given classes of input states, taking into account imperfections of the interferometer.

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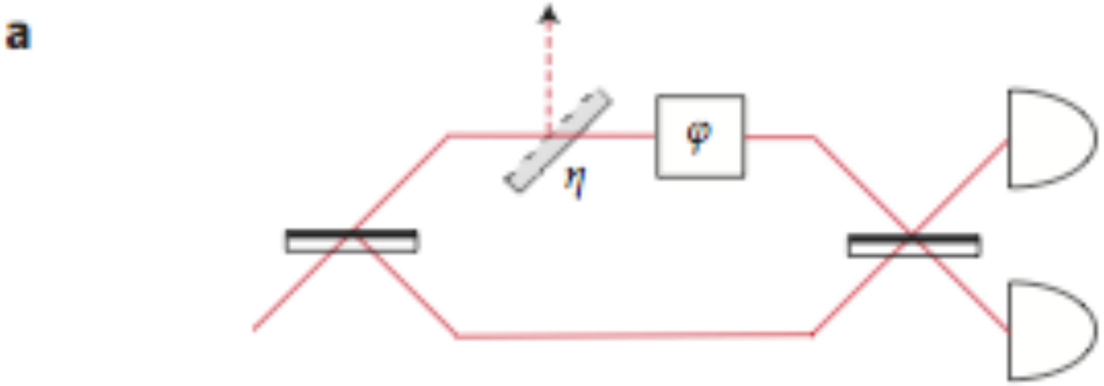

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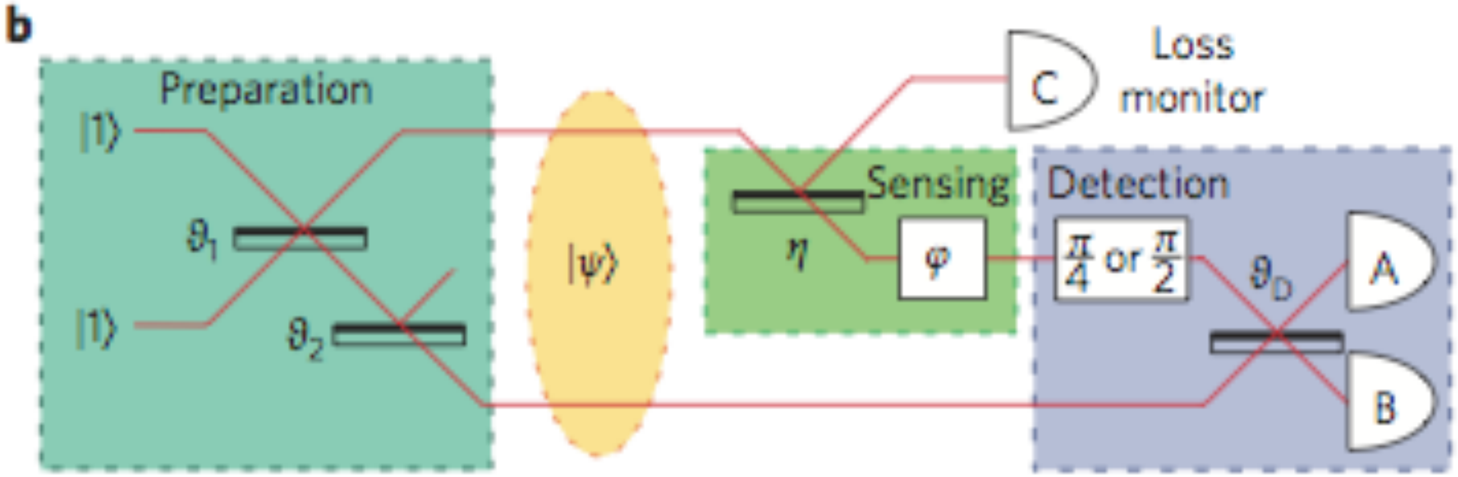

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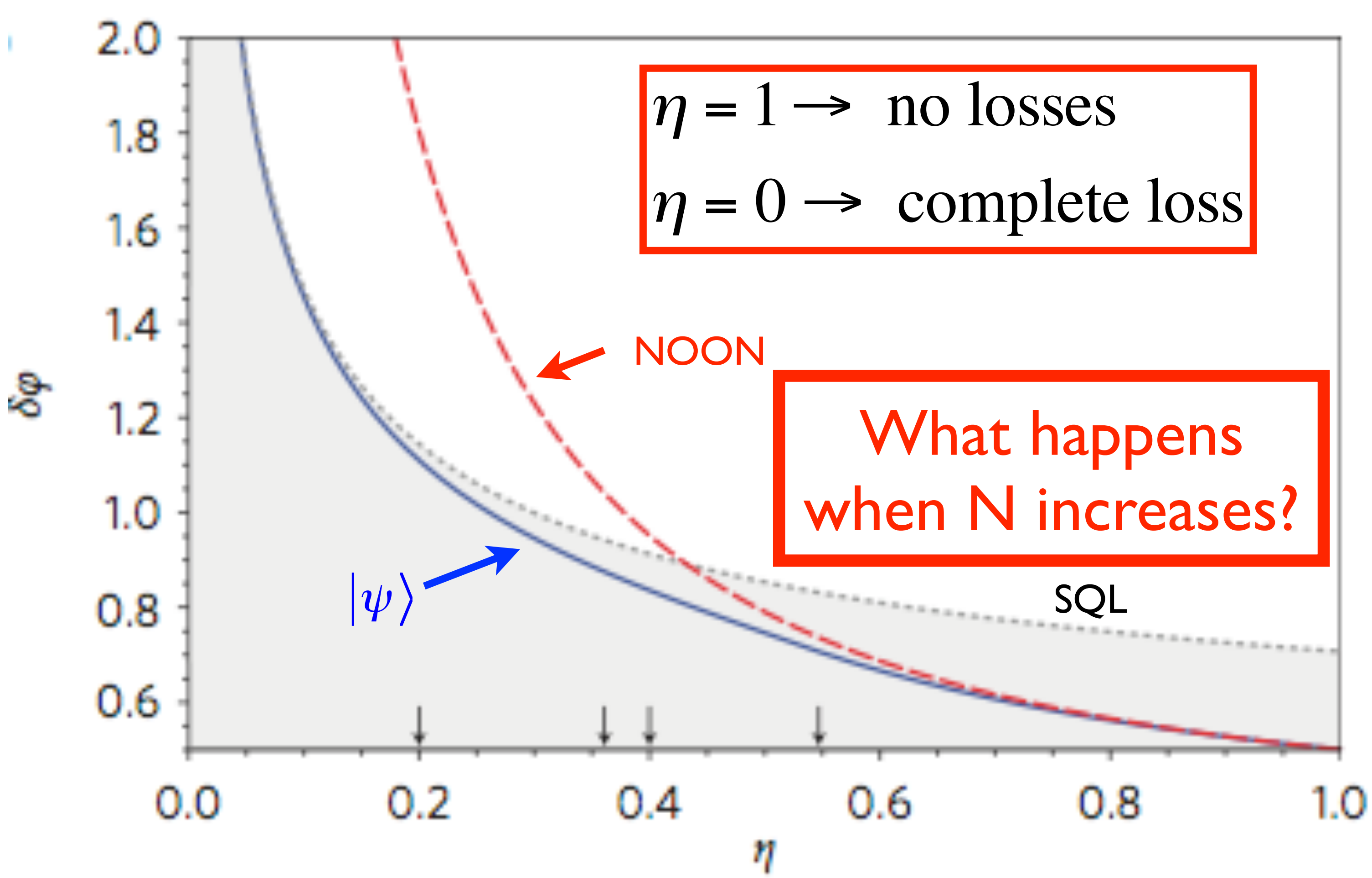

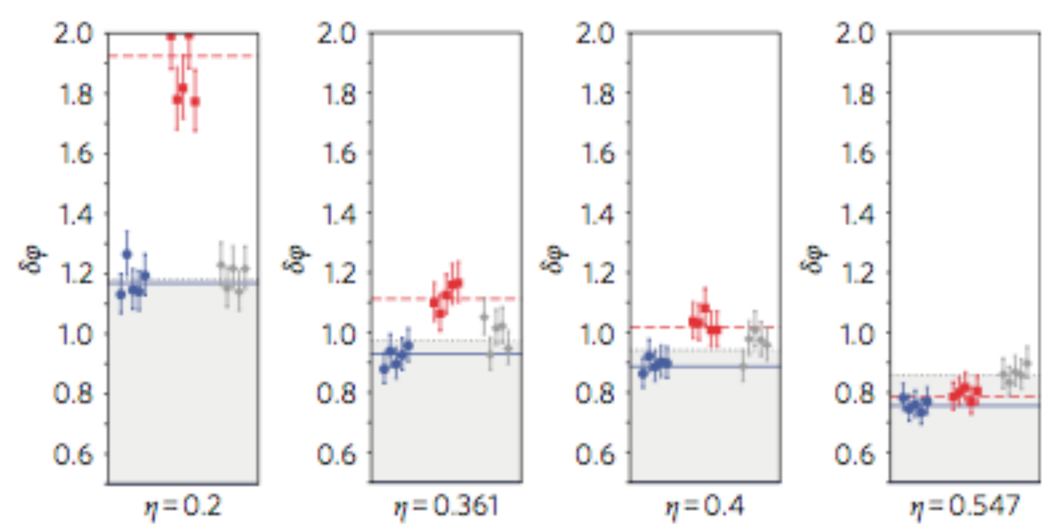

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Open-system evolution and quantum channels

The evolution of an open system can be described by the Hamiltonian

$$H = H_S \otimes \mathbf{1}_E + \mathbf{1}_S \otimes H_E + V_{SE}$$

H_S and H_E ---> free-evolution Hamiltonians of system and environment

V_{SE} ---> interaction between the two parties. Effective time evolution of S:

$$\rho_S(t) = \mathrm{Tr}_E\left[\rho_{SE}(t)\right]$$

Assuming that initially S and E are not correlated, and that the initial state of the environment is $|0\rangle_E$, then $\rho_{SE}(0) = \rho_S^{\text{in}} \otimes |0\rangle_E\langle 0|$ and

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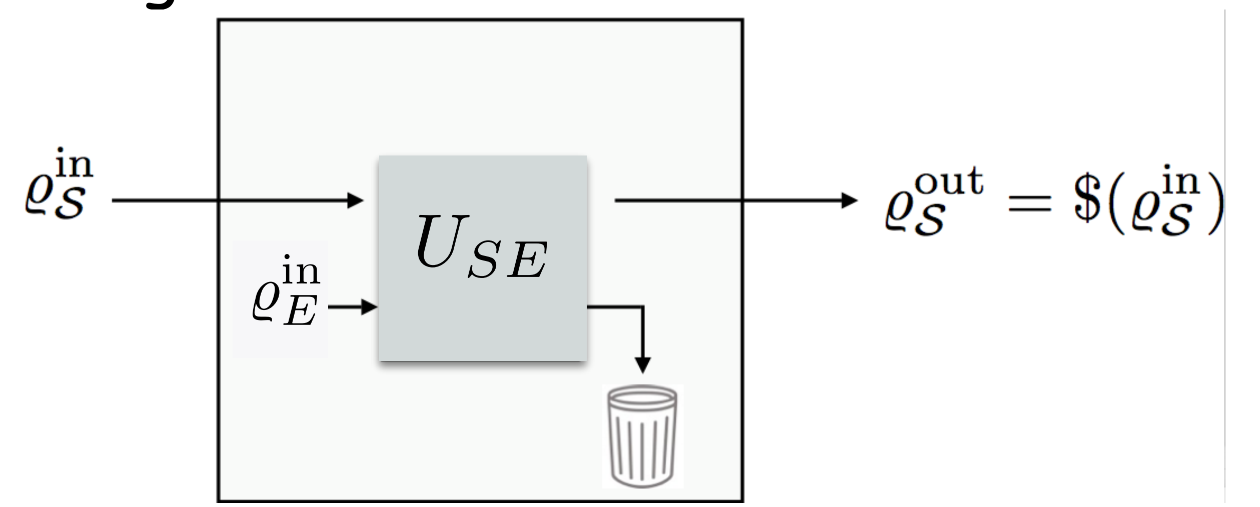
$$\rho_S^{\mathrm{out}} = \mathrm{Tr}_E\left[U_{SE}\left(\rho_S^{\mathrm{in}} \otimes |0\rangle_E\langle 0|\right) U_{SE}^\dagger\right]$$
$$= \sum_\mu {}_E\langle \mu|U_{SE}|0\rangle_E\, \rho_S^{\mathrm{in}}\; {}_E\langle 0|U_{SE}^\dagger|\mu\rangle_E$$
$$= \sum_\mu K_\mu \rho_S^{\mathrm{in}} K_\mu^\dagger = \$(\rho_S^{\mathrm{in}})$$



where $\{|\mu\rangle\}$ is a basis of E, and $K_\mu \equiv {}_E\langle \mu|U_{SE}|0\rangle_E$ are the Kraus operators (this is the Kraus decomposition of a quantum channel).

Differential form of this evolution ---> master equation for the reduced density matrix of the system

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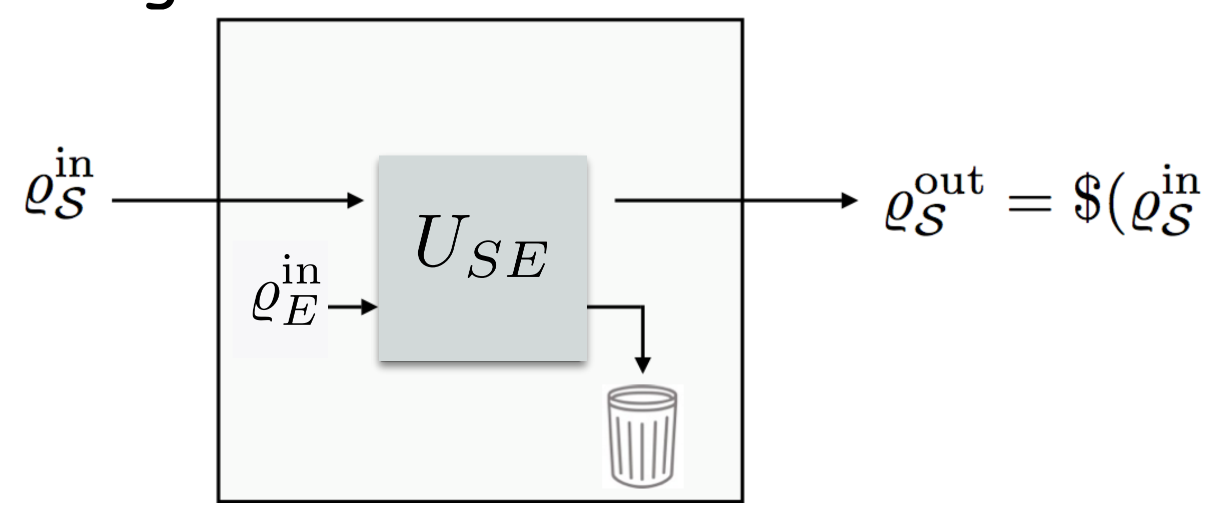

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Purification of an evolution

Given the Kraus decomposition of a quantum channel, it is possible to find a correspondent unitary evolution of the system plus an environment.

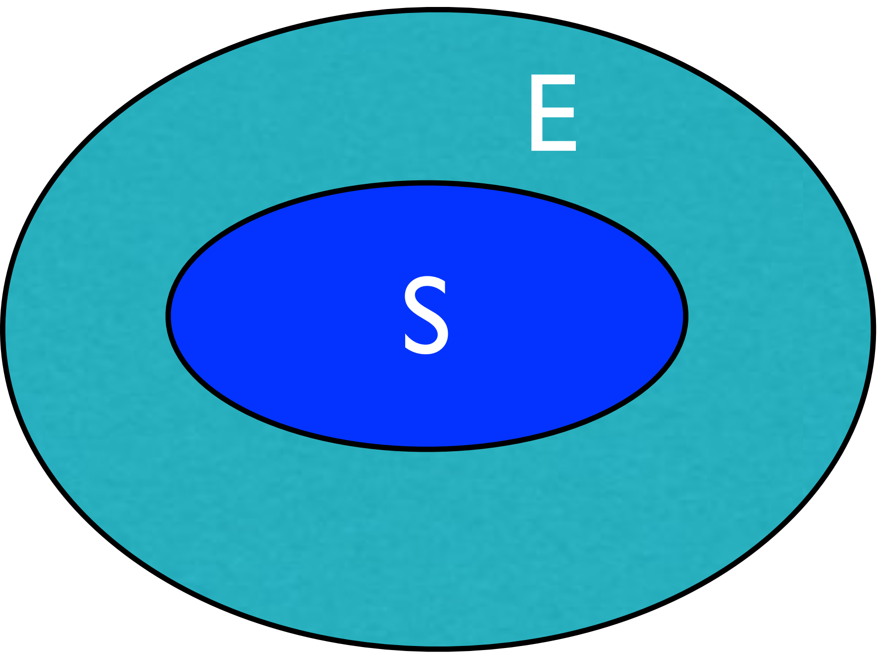
This unitary evolution is not necessarily the same as the one derived from the original Hamiltonian: the "effective" environment may be different than the real environment E, but it leads however to the same dynamics for all the states in S.

We shall use this purification strategy in order to develop a general framework for the estimation of parameters in noisy quantum-enhanced metrology.

Parameter estimation in open systems: Extended space approach

B. M. Escher, R. L. Matos Filho, and L. D., **Nature Physics** 7, 406 (2011); **Braz. J. Phys.** 41, 229 (2011)

Given initial state and non-unitary evolution, define in S+E



$|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S |0\rangle_E$ (Purification)

Then

$$\mathscr{F}_Q \equiv \max_{\hat{E}_j^{(S)}\otimes\hat{1}} F\left(\hat{E}_j^{(S)}\otimes\hat{1}\right) \leq \max_{\hat{E}_j^{(S,E)}} F\left(\hat{E}_j^{(S,E)}\right) \equiv \mathscr{C}_Q$$

since measurements on S+E should yield more information than measurements on S alone.

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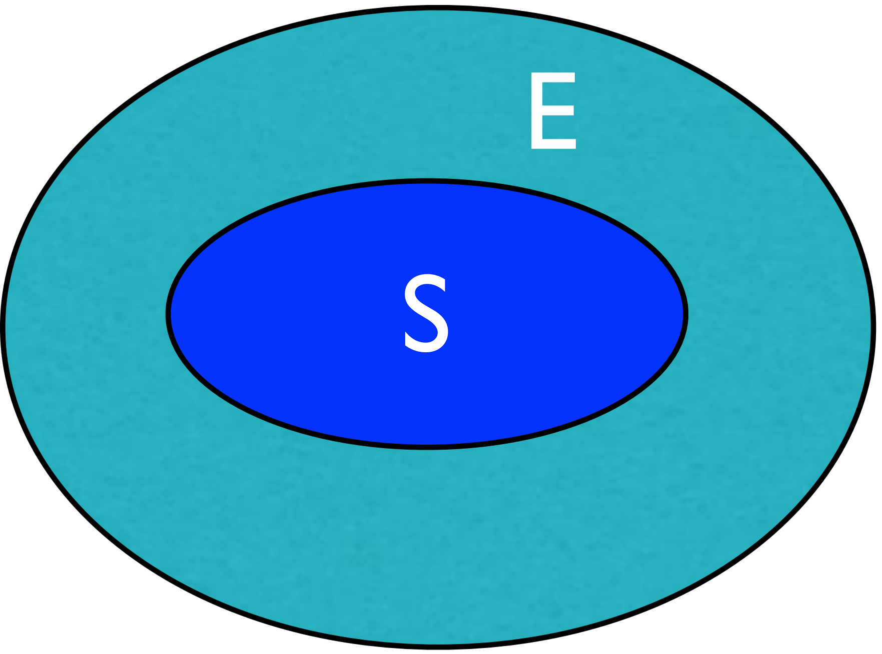

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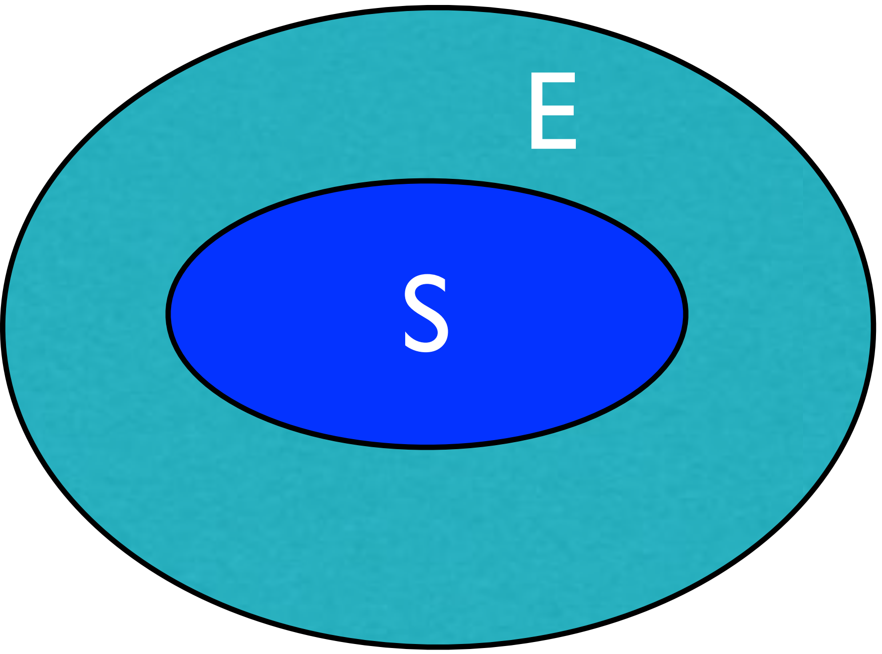

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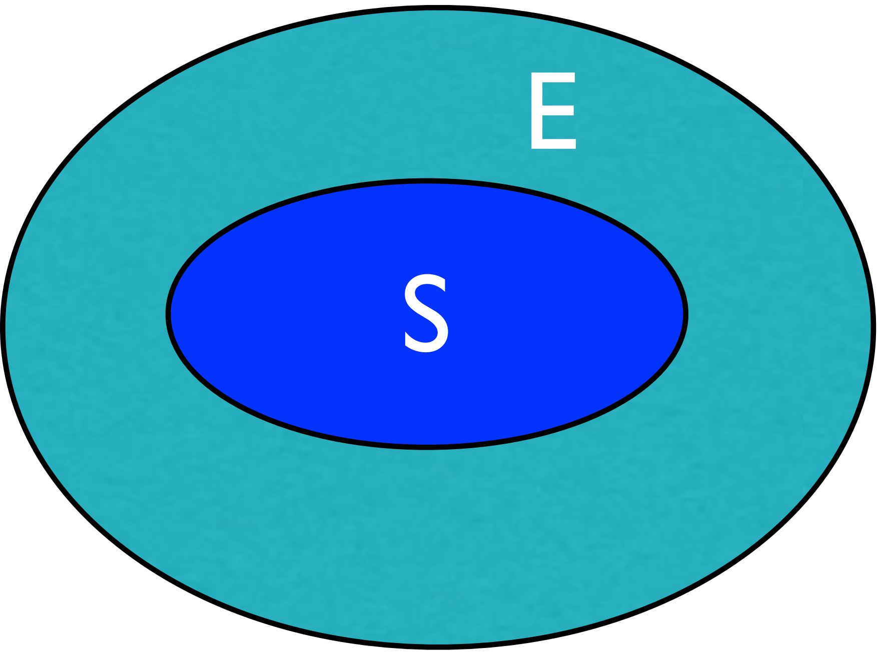

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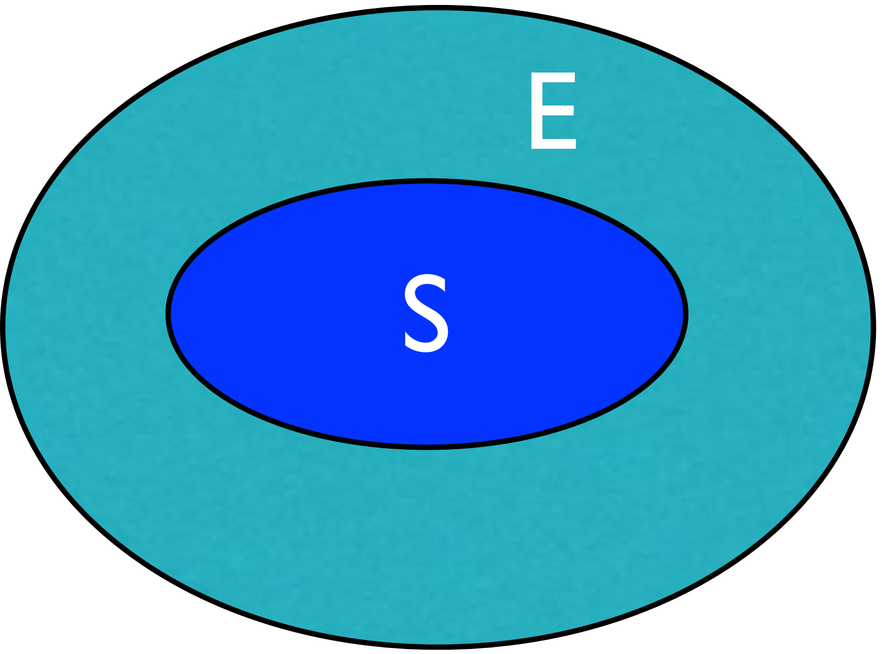

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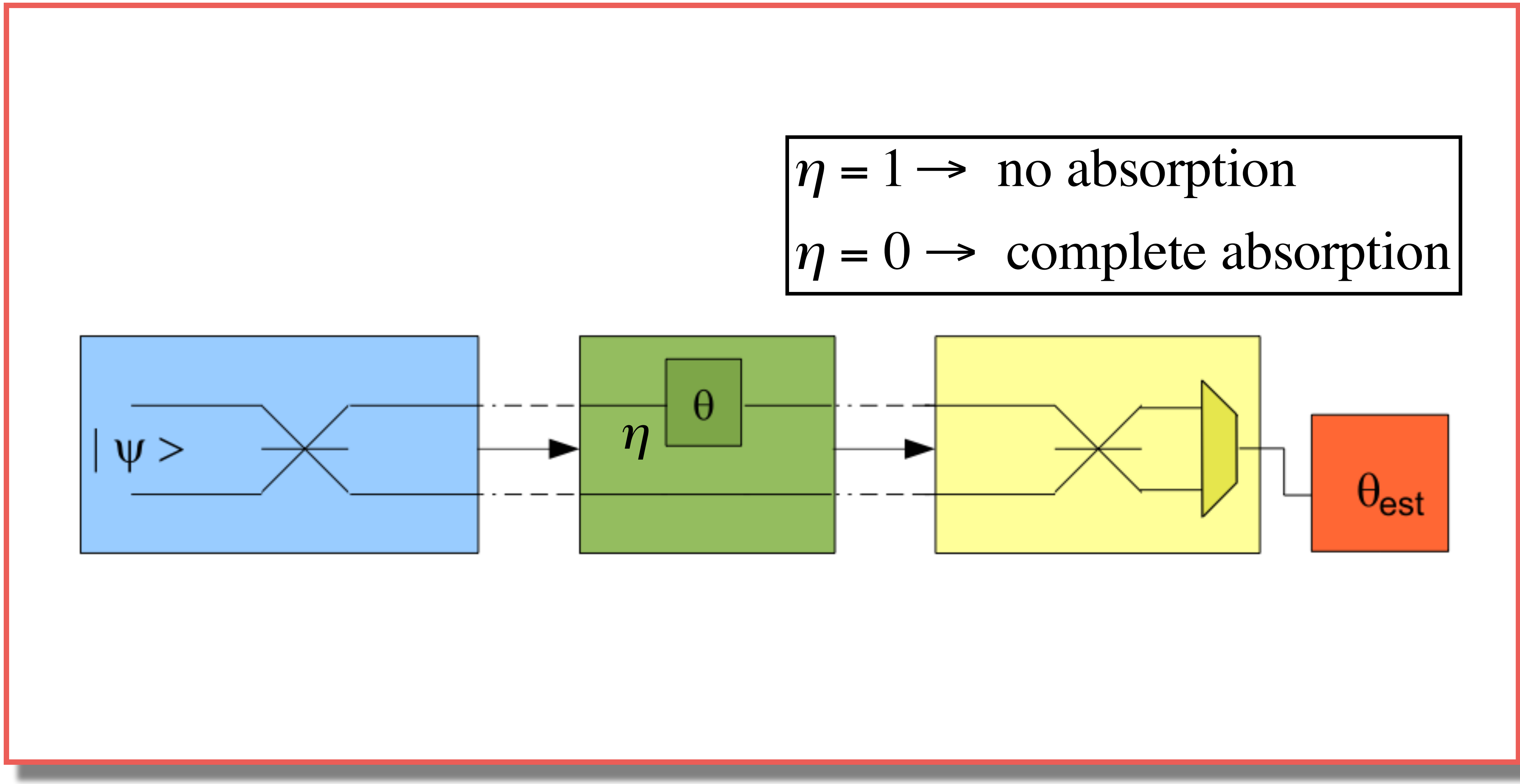
Then, monitoring S+E yields same information as monitoring S

Quantum limits for lossy optical interferometry



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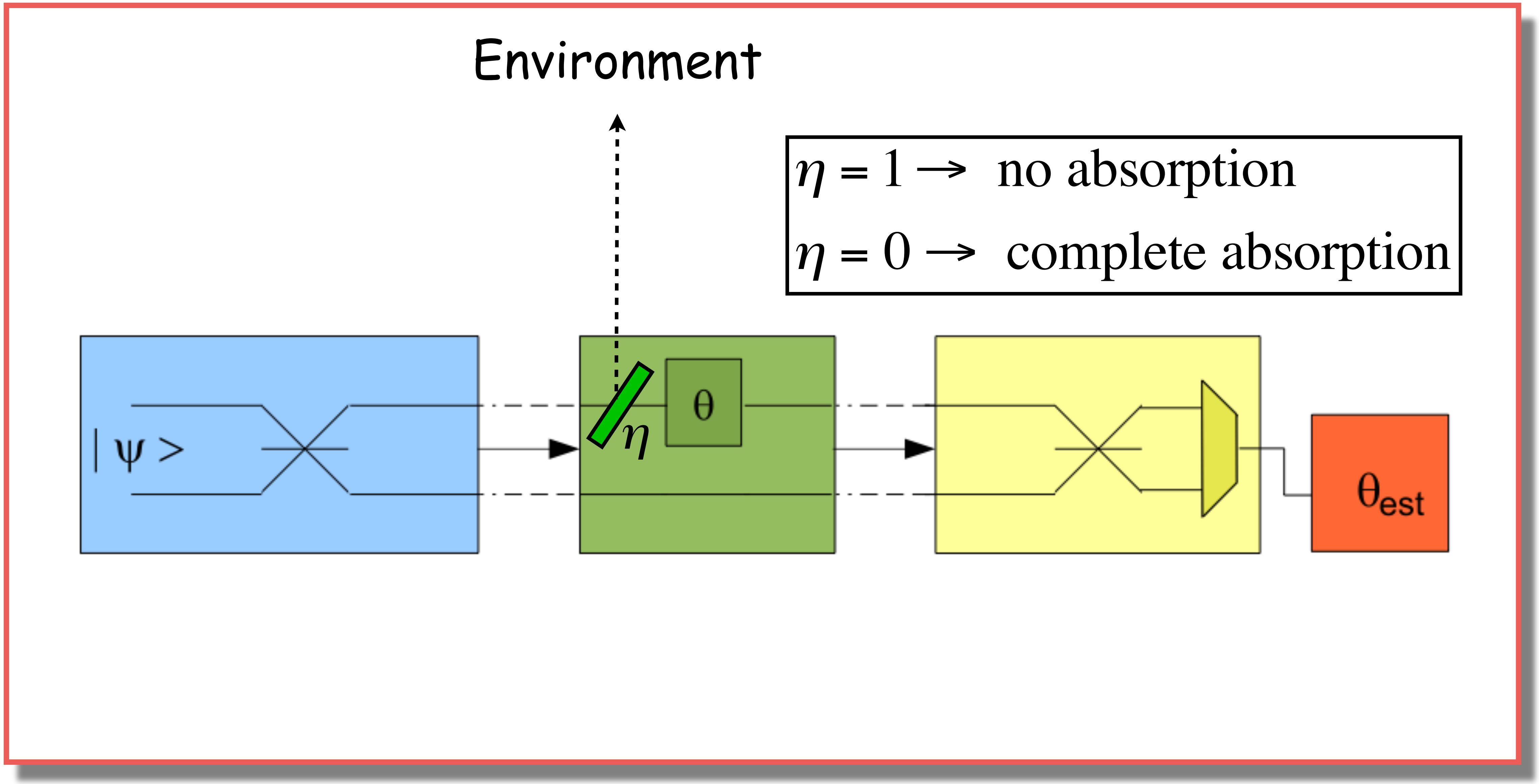
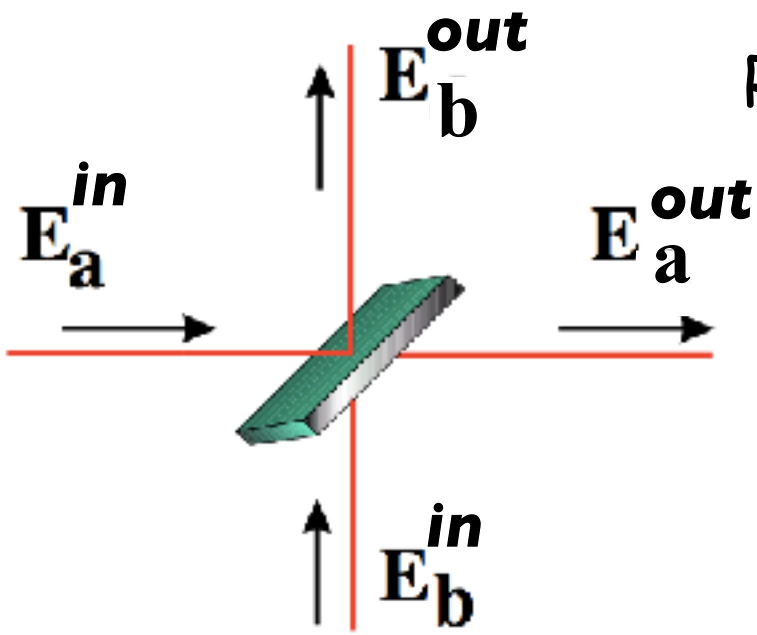


Remarks on beam splitters



Relation between positive-frequency parts of the fields:

$$\begin{pmatrix} E_a^{out} \\ E_b^{out} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} E_a^{in} \\ E_b^{in} \end{pmatrix},$$

Conservation of energy: $\left|E_a^{out}\right|^2 + \left|E_b^{out}\right|^2 = \left|E_a^{in}\right|^2 + \left|E_b^{in}\right|^2$

Quantum beam splitter:

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix}$$ (Heisenberg operators)

where the operator â annihilates photons in mode a: $\hat{a}|N\rangle = \sqrt{N}|N-1\rangle$ and $|N\rangle$ is the Fock state with N photons, with $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle$, where $\hat{a}^\dagger\hat{a}$ is the number operator.

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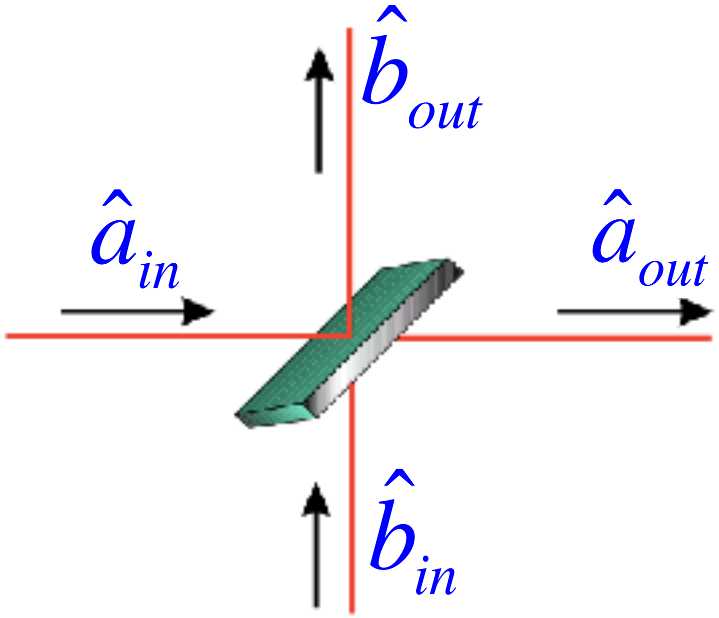


$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix}$$

Exercises:

1. Energy conservation: Show that $\hat{a}^{\dagger}_{out}\hat{a}_{out} + \hat{b}^{\dagger}_{out}\hat{b}_{out} = \hat{a}^{\dagger}_{in}\hat{a}_{in} + \hat{b}^{\dagger}_{in}\hat{b}_{in}$

2. Beam-splitter operator: Show that, if $\hat{U}_B(\theta) = \exp\left[-i\theta\left(\hat{a}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}\right)/2\right]$, then

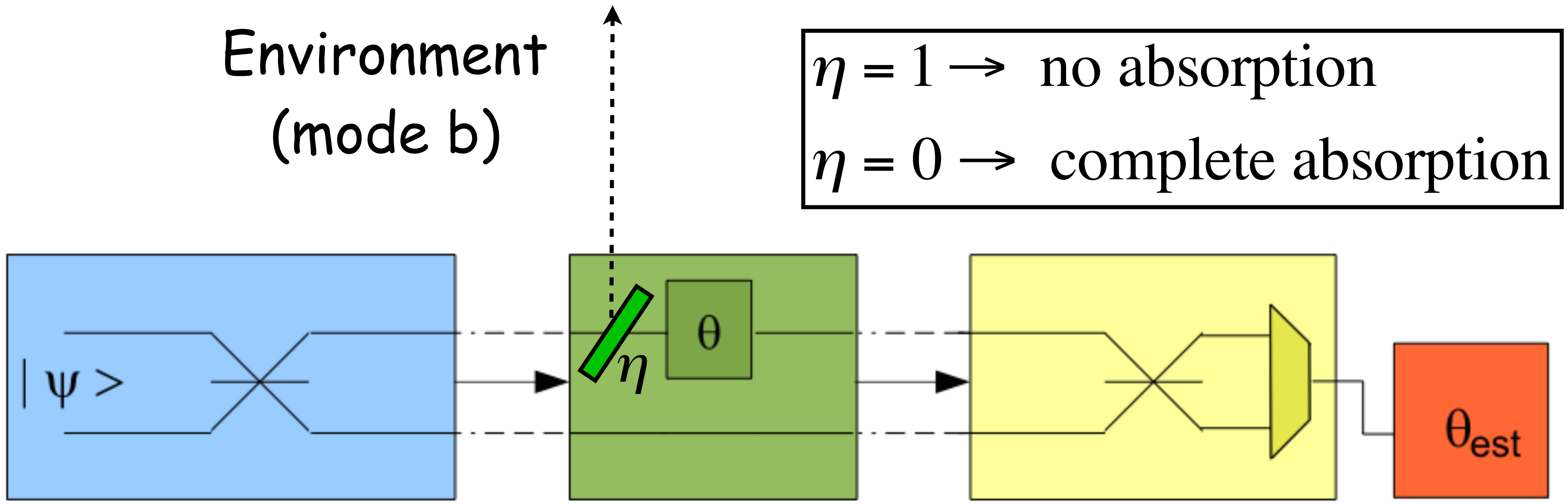
$$\hat{U}^{\dagger}_B(\theta)\hat{a}\hat{U}_B(\theta) = \hat{a}\cos(\theta/2) - i\hat{b}\sin(\theta/2) = \hat{a}_{out}$$

$$\hat{U}^{\dagger}_B(\theta)\hat{b}\hat{U}_B(\theta) = -i\hat{a}\cos(\theta/2) + \hat{b}\sin(\theta/2) = \hat{b}_{out}$$

In terms of the transmissivity $\eta = \cos(\theta/2)$:

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = \begin{pmatrix} \sqrt{\eta} & -i\sqrt{1-\eta} \\ -i\sqrt{1-\eta} & \sqrt{\eta} \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix} \Rightarrow \boxed{\hat{U}_B(\theta) = \exp\left[-i\arccos\left(\sqrt{\eta}\right)\left(\hat{a}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}\right)\right]}$$

Quantum limits for lossy optical interferometry



Possible state for environment E (mode b) and system S (mode a):

$$\left|\psi(\theta)\right\rangle_{SE} = e^{i\theta\hat{n}_S}\hat{U}_B\left(\sqrt{\eta}\right)\left|\psi_0\right\rangle_S\left|0\right\rangle_E$$

This is one of many possible purifications. To get a purification that leads to a final state of E with less information on θ, one possibility is to apply to E the operator $\exp\left(-i\alpha\theta\hat{n}_E\right)$, with $\hat{n}_E$ being the number of photons in the environment mode:

$$\left|\psi(\theta)\right\rangle_{SE} = e^{-i\alpha\theta\hat{n}_E}e^{i\theta\hat{n}_S}\hat{U}_B\left(\sqrt{\eta}\right)\left|\psi_0\right\rangle_S\left|0\right\rangle_E$$

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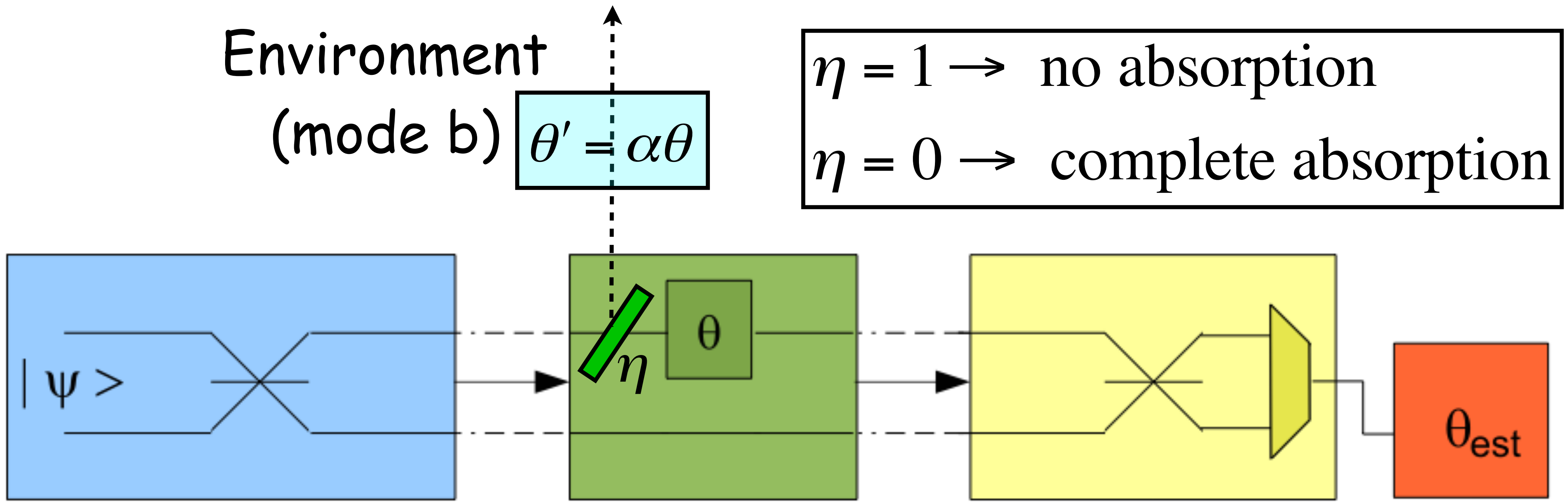

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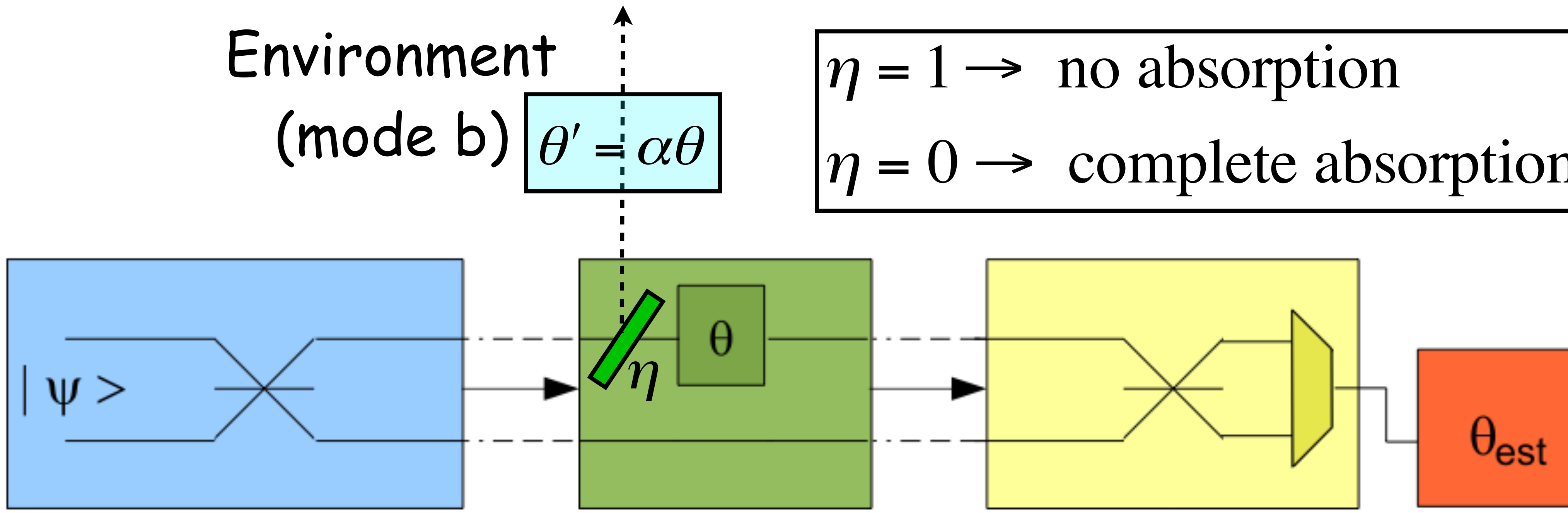

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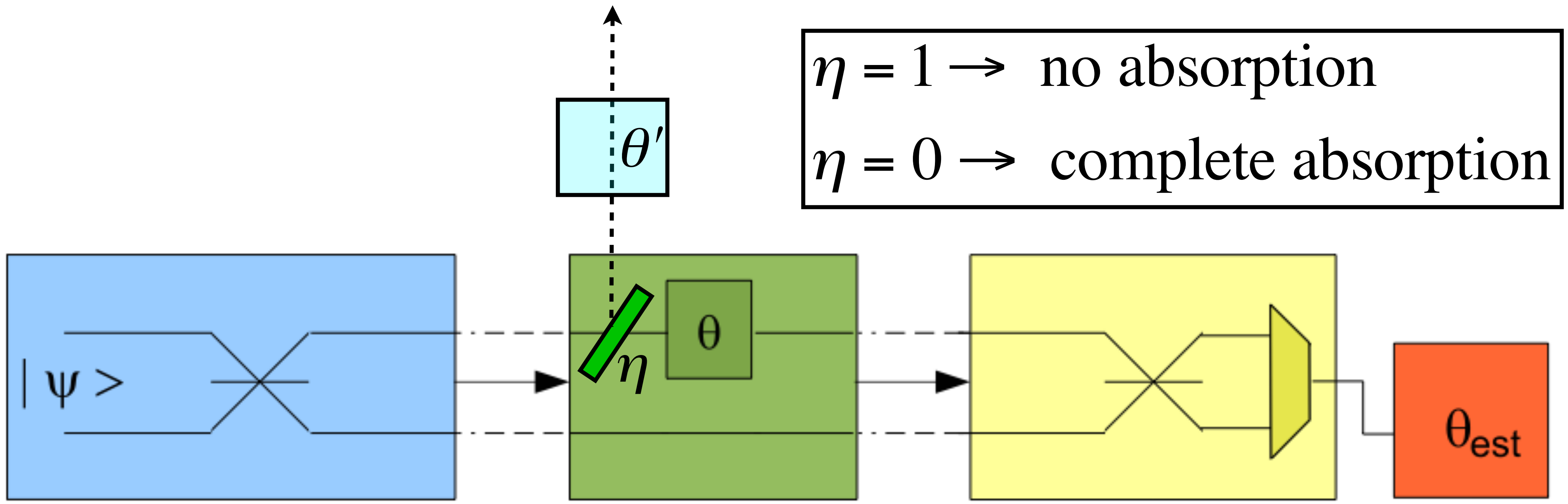
Reduced evolution is not changed by the extra unitary transformation

Quantum limits for lossy optical interferometry

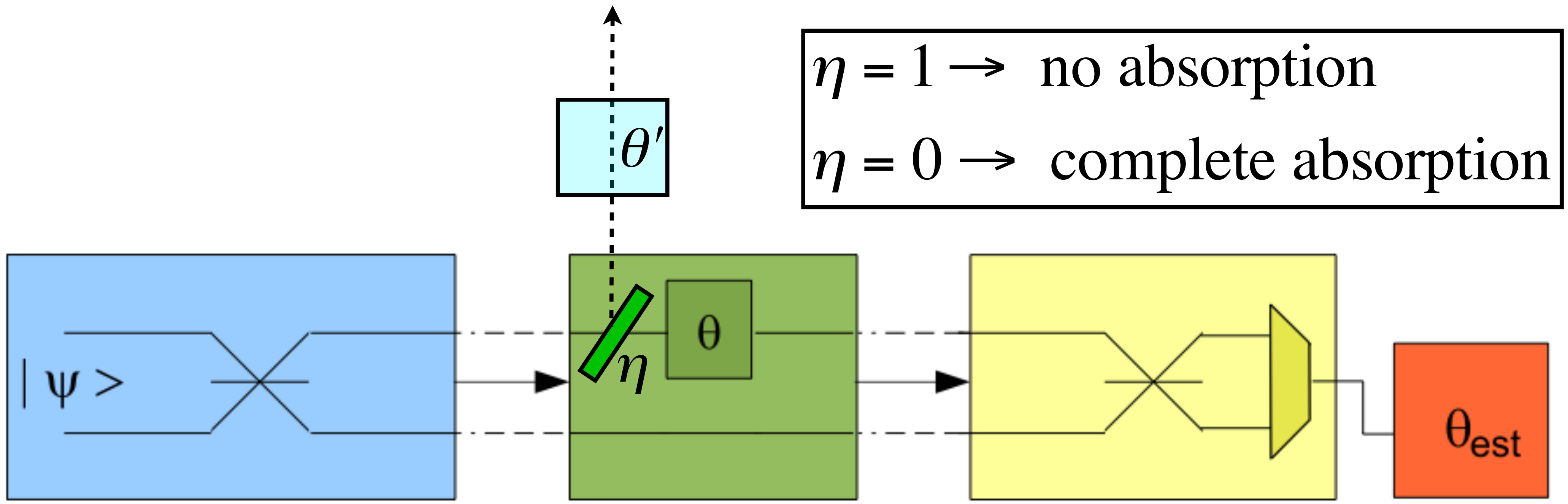


The quantum Fisher information corresponding to this evolution is

$$\mathcal{F}_Q\left[\theta, |\psi(\theta,\alpha)\rangle_{SE}\right] = 4\, {}_S\langle\psi_0|_E\langle 0|\Delta\hat{H}^2|0\rangle_E|\psi_0\rangle_S$$

where $\hat{H}(\alpha,\theta) = i\dfrac{d}{d\theta}\left[\hat{U}_B^{\dagger} e^{-i\theta\hat{n}_s} e^{i\alpha\theta\hat{n}_E}\right] e^{-i\alpha\theta\hat{n}_E} e^{i\theta\hat{n}_s}\hat{U}_B$

Quantum limits for lossy optical interferometry



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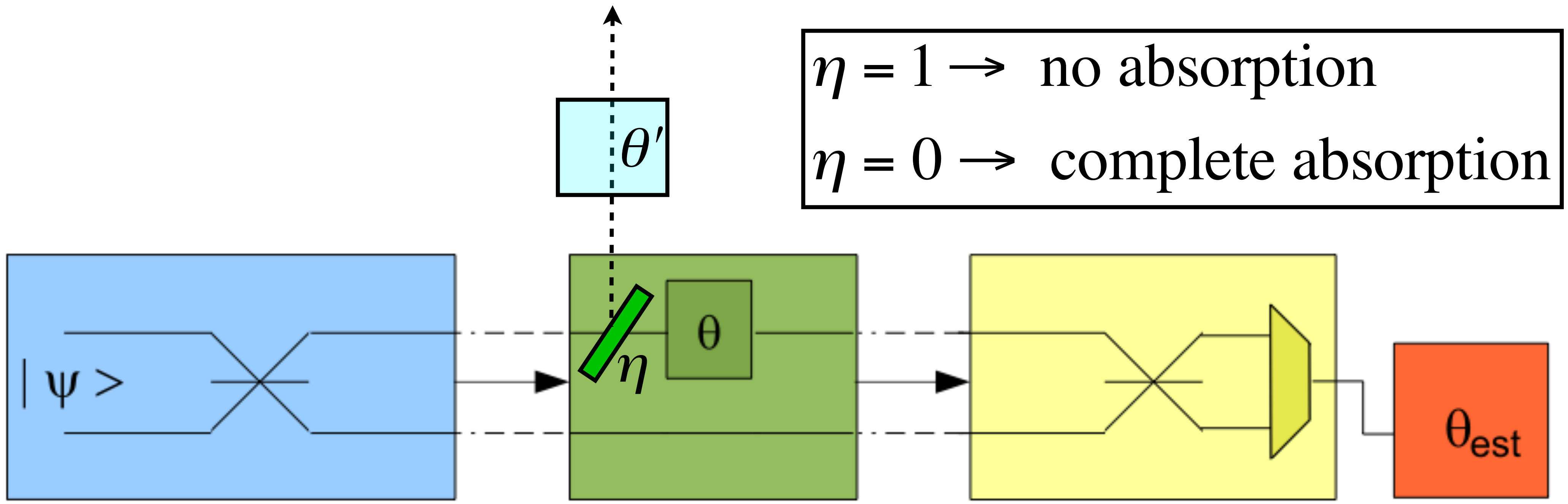
Minimization of the quantum Fisher information of system + environment yields an upper bound for the Fisher information of the system:

$$\mathcal{C}_Q(\hat{\rho}_0) = \frac{4\eta\langle\hat{n}\rangle_0\Delta^2\hat{n}_0}{(1-\eta)\Delta^2\hat{n}_0 + \eta\langle\hat{n}\rangle_0}$$

where $\langle\hat{n}\rangle_0 = {}_S\langle\psi_0|\hat{n}_S|\psi_0\rangle_S$

$\Delta^2 n_0 = {}_S\langle\psi_0|(\Delta\hat{n}_S)^2|\psi_0\rangle_S$

Quantum limits for lossy optical interferometry



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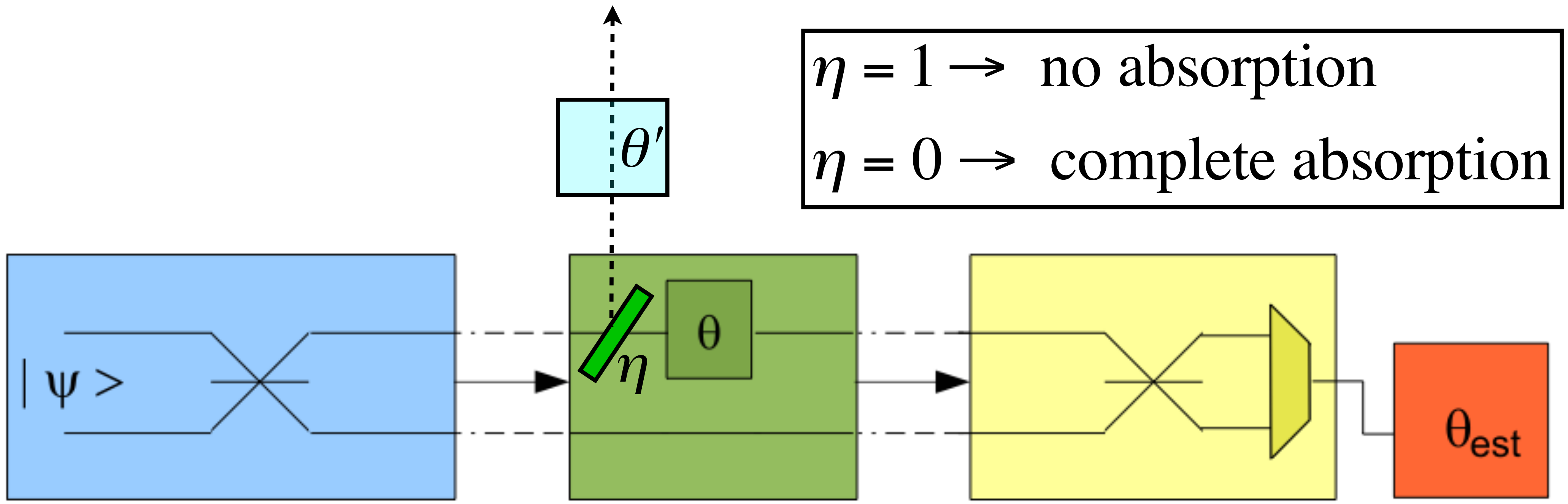
Quantum limits for lossy optical interferometry



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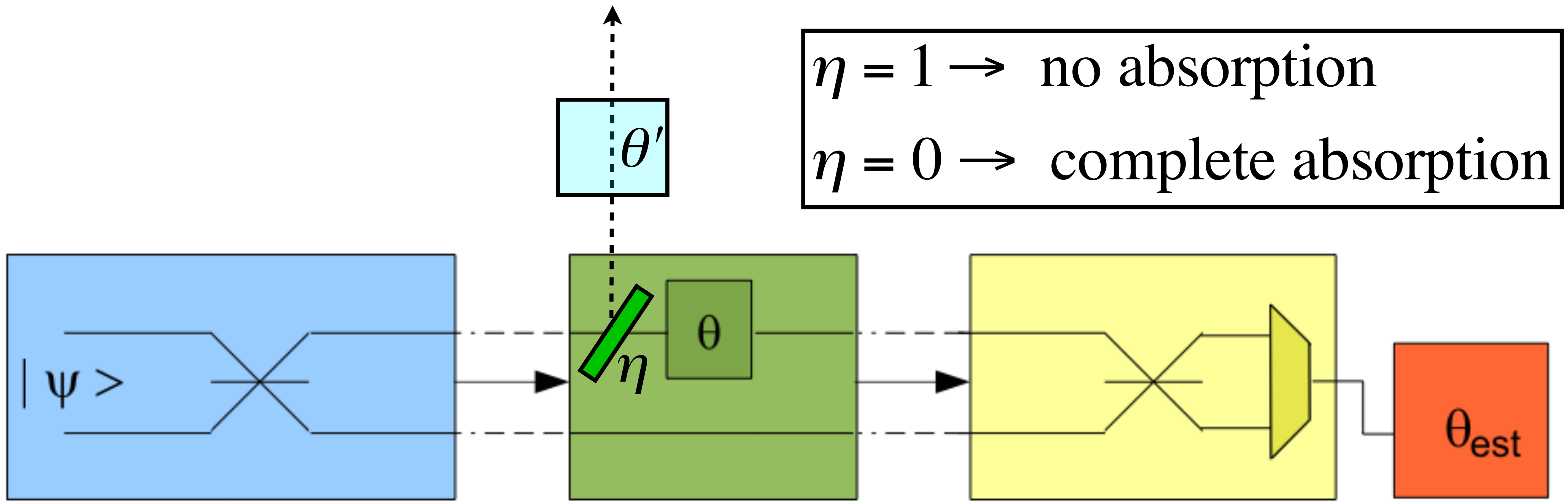
Low-dissipation limit: $(1-\eta)\Delta^2\hat{n}_0 \ll \eta\langle\hat{n}\rangle_0 \Longrightarrow \mathcal{C}_Q \rightarrow 4\Delta^2\hat{n}_0$

(noiseless limit)

High-dissipation limit: $(1-\eta)\Delta^2\hat{n}_0 \gg \eta\langle\hat{n}\rangle_0 \Longrightarrow \delta\theta \geq \sqrt{(1-\eta)/4\eta\langle\hat{n}\rangle_0}$

(shot-noise scalling)

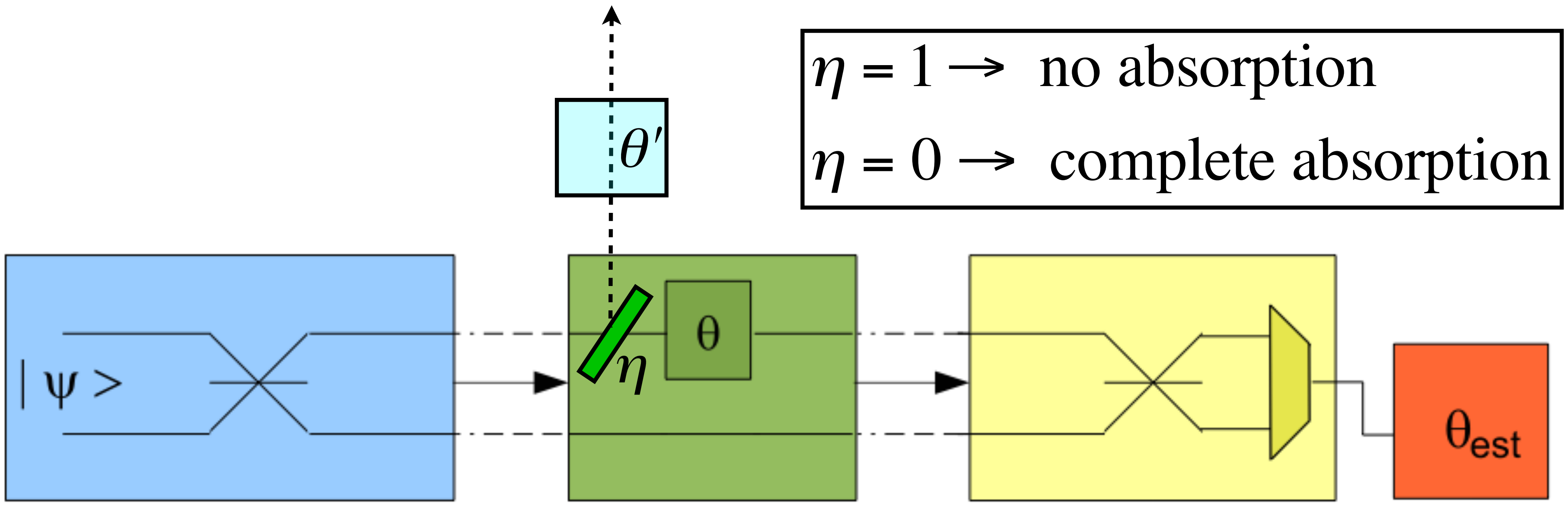
Quantum limits for lossy optical interferometry



$\eta = 1 \rightarrow$ no absorption

$\eta = 0 \rightarrow$ complete absorption

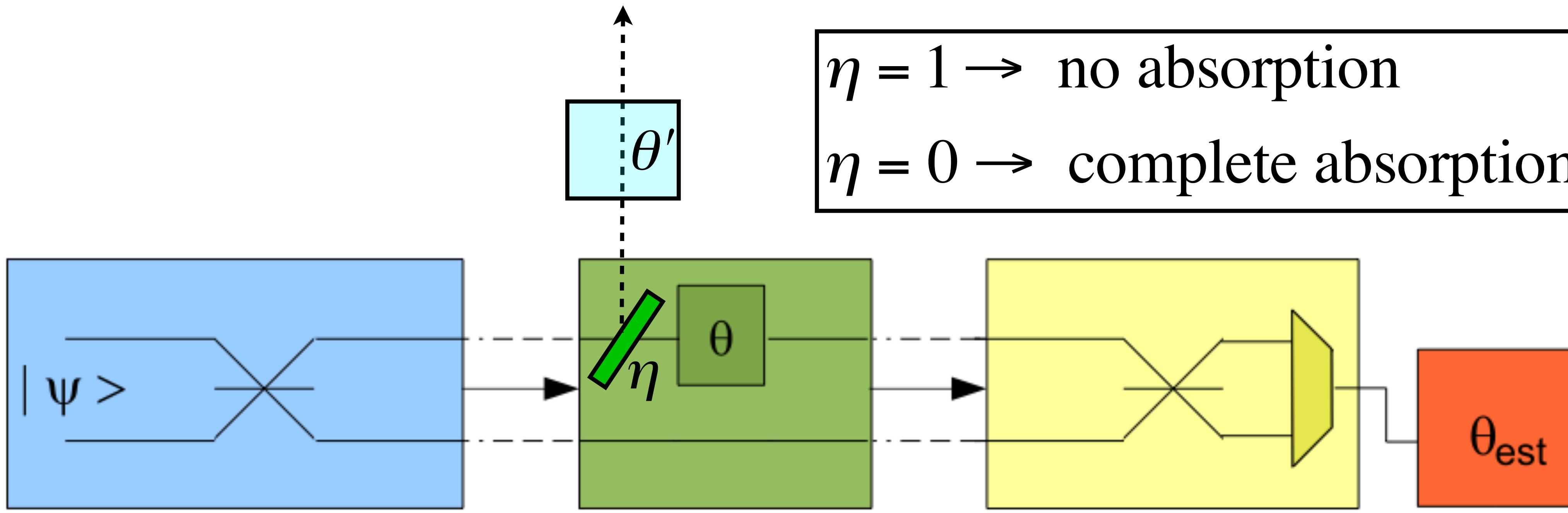
Quantum limits for lossy optical interferometry



States with well-defined total photon number:

$$|\psi_0\rangle = \sum_{n=0}^{N} \beta_n |n, N-n\rangle$$

Quantum limits for lossy optical interferometry



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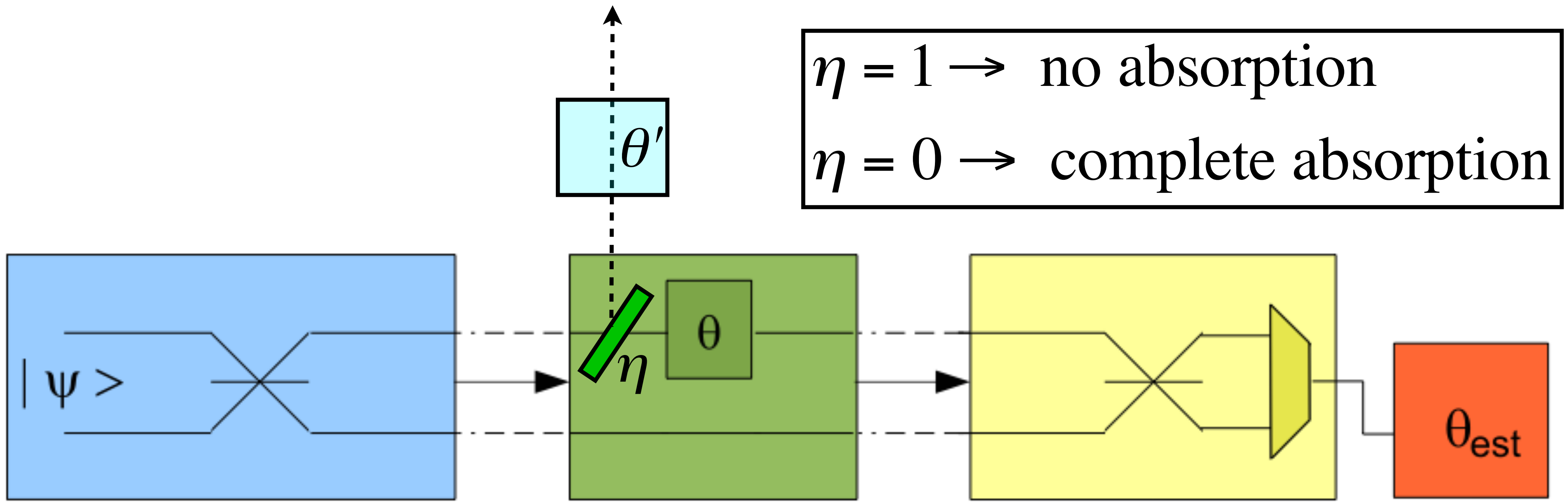
Quantum limits for lossy optical interferometry



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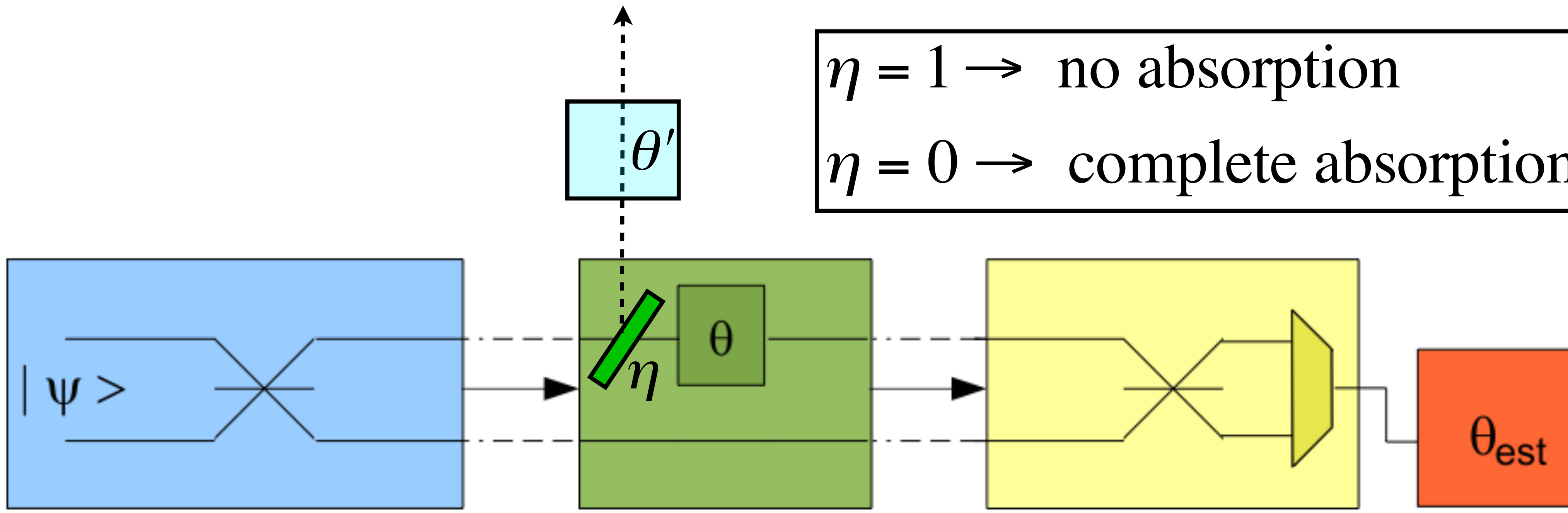
For N sufficiently large, $1/\sqrt{N}$ behavior is always reached!

Quantum limits for lossy optical interferometry



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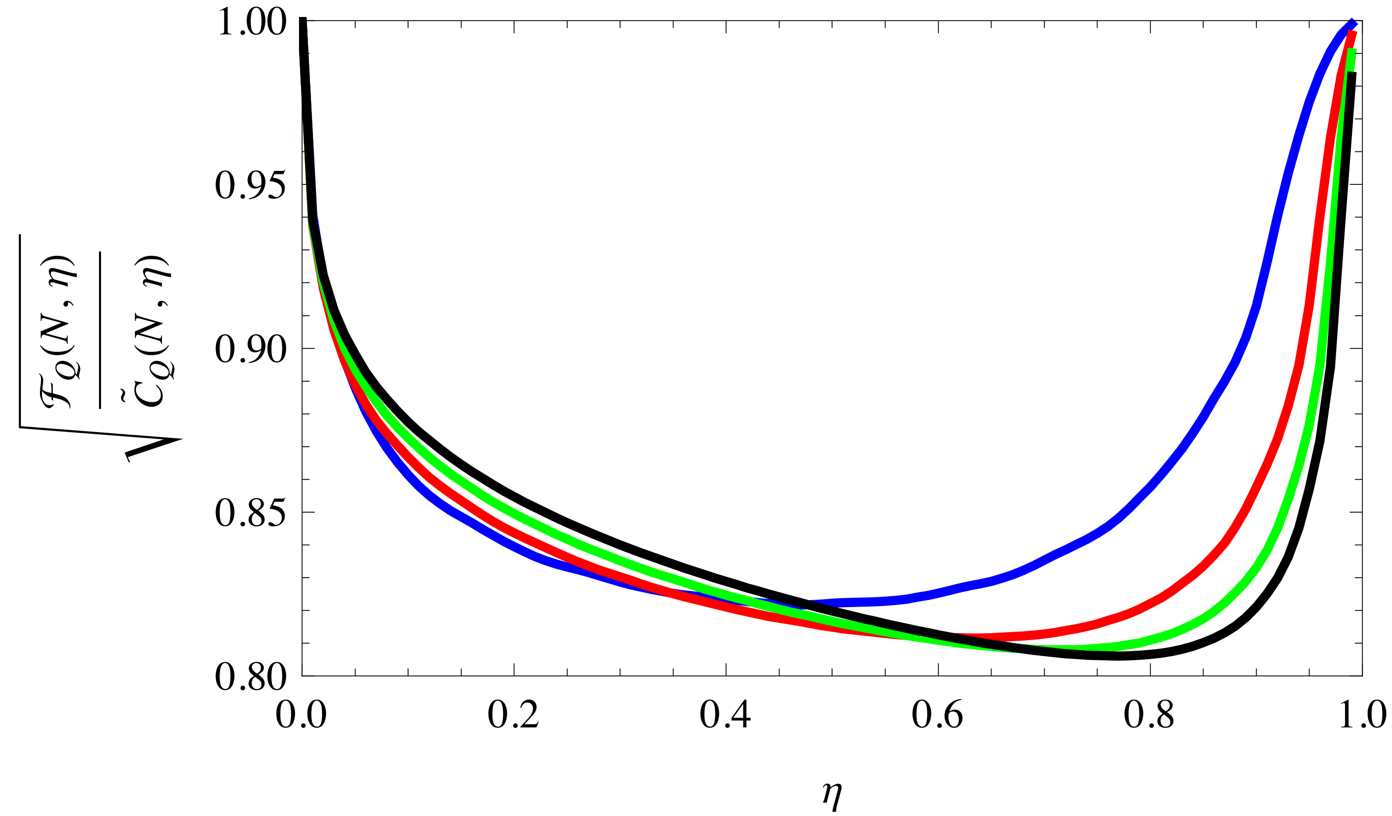
$N \ll \frac{\eta}{1-\eta} \Rightarrow \sqrt{\nu}\delta\theta \geq 1/N$ —>Heisenberg scaling

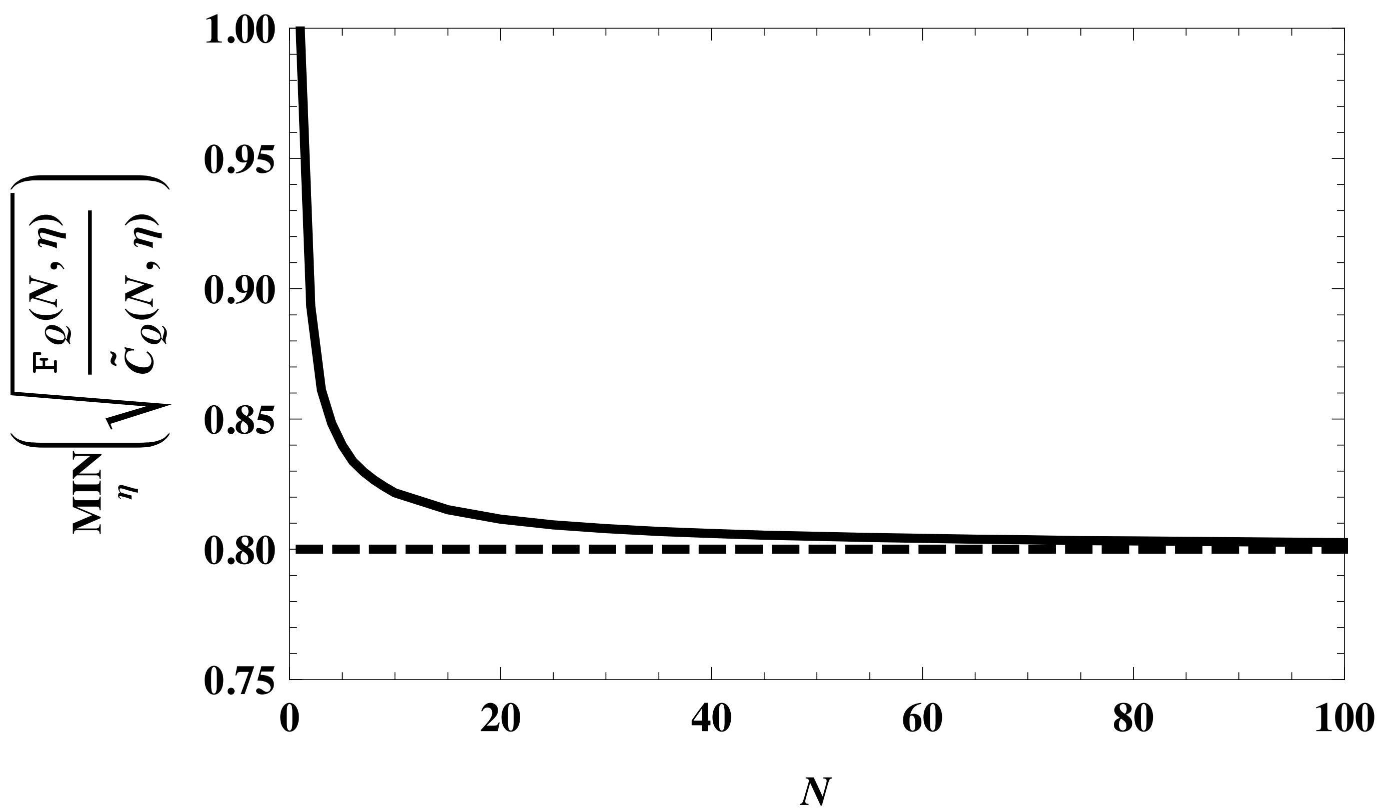
$N \gg \frac{\eta}{1-\eta} \Rightarrow \delta\theta \geq \frac{\sqrt{1-\eta}}{2\sqrt{\nu\eta N}}$ —>Standard scaling

For N sufficiently large, $1/\sqrt{N}$ behavior is always reached!

How good is this bound?



Comparison between the numerical maximum value of $\mathscr{F}_Q$ and the upper bound $\mathscr{C}_Q$ as a function of η, for $N = 10$ (blue), $N = 20$ (red), $N = 30$ (green), and $N = 40$ (black).



Behavior of the minimum for all values of η, as a function of N

Phase diffusion in optical interferometer

PRL 109, 190404 (2012) PHYSICAL REVIEW LETTERS week ending 9 NOVEMBER 2012

Quantum Metrological Limits via a Variational Approach

B. M. Escher,* L. Davidovich, N. Zagury, and R. L. de Matos Filho
Instituto de Física, Universidade Federal do Rio de Janeiro, 21.941-972, Rio de Janeiro (RJ) Brazil
(Received 29 June 2012; published 9 November 2012)

$$\dot{\rho} = \Gamma \mathcal{L}[a^\dagger a]\rho, \quad \mathcal{L}[O]\rho = 2O\rho O^\dagger - O^\dagger O\rho - \rho O^\dagger O$$

$$\Rightarrow \rho(t) = \sum_{m.n} e^{-\beta^2 (n-m)^2} \rho_{n,m}(0)|n\rangle\langle m|, \quad \beta = \Gamma t$$

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Possible purification:

Radiation pressure

Ground state of mirror (harmonic oscillator)

$$|\Phi_{S,E}(\phi)\rangle = e^{-i\phi\hat n_S} e^{i(2\beta)\hat n_S \hat x_E}|\psi_S\rangle|0_E\rangle$$

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Choose instead: $$|\Phi_{S,E}(\phi)\rangle = e^{i\phi\lambda\hat p_{\mathrm{E}}/(2\beta)} e^{-i\phi\hat n_S} e^{i(2\beta)\hat n_S \hat x_{\mathrm{E}}}|\psi_S\rangle|0_E\rangle$$

$\lambda \longrightarrow$ Variational parameter

Phase diffusion in optical interferometer

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$$\Rightarrow C_Q = (1-\lambda)^2 4\Delta n^2 + \lambda^2/(2\beta^2)$$

$\lambda \longrightarrow$ Variational parameter

Phase diffusion in optical interferometer

Phase diffusion in optical interferometer

$$\delta\phi_{pd} \geq \sqrt{\frac{1}{\nu}\left(\frac{1}{4\Delta n^2} + 2\beta^2\right)}$$

Intrinsic quantum feature

Phase diffusion

Very close to numerical value obtained by Genoni, Olivares, and Paris for Gaussian state - PRL 106, 153603 (2011)

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For Gaussian states:

$$\Delta n^2 \leq 2N(N+1)$$

(N is the average photon number)

Then:

$$C_Q^{\text{opt}} \leq C_Q^{\max} \equiv \left[2\beta^2 + \frac{1}{8N(N+1)}\right]^{-1}$$

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Comparison with numerical results

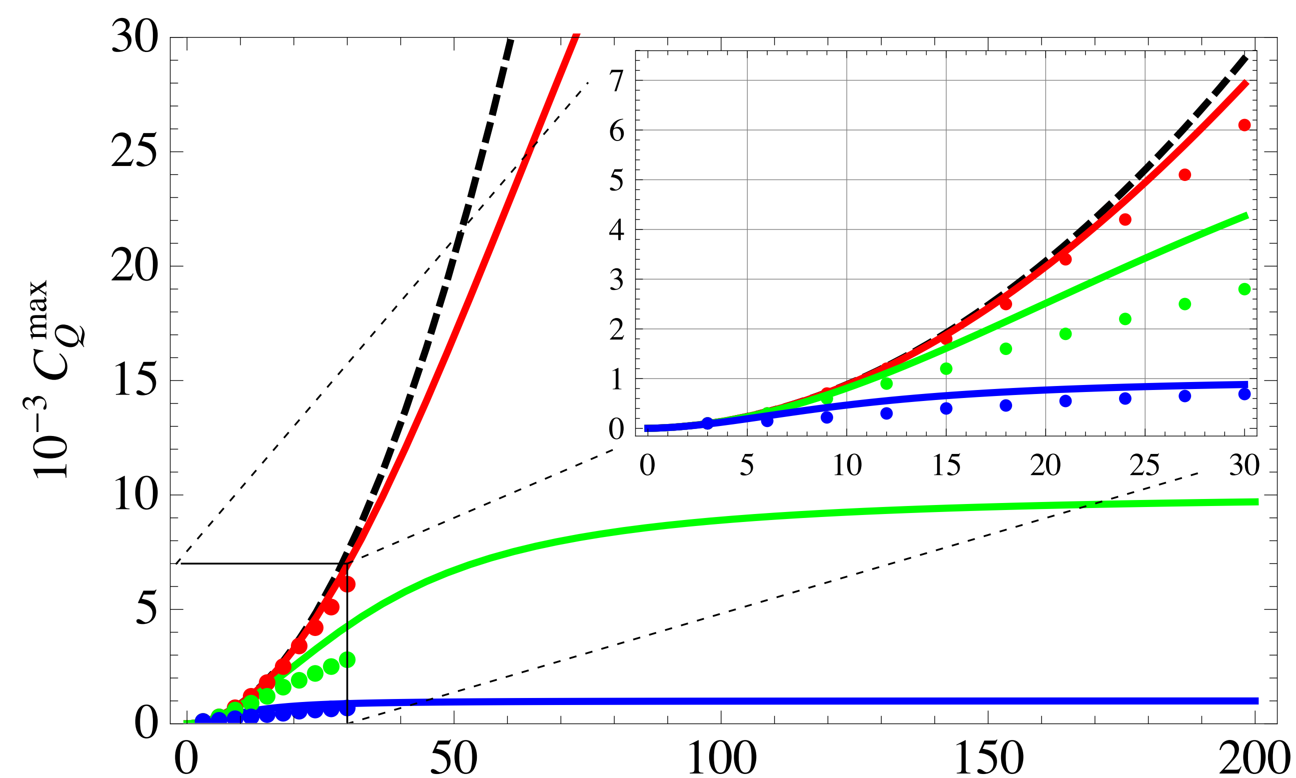
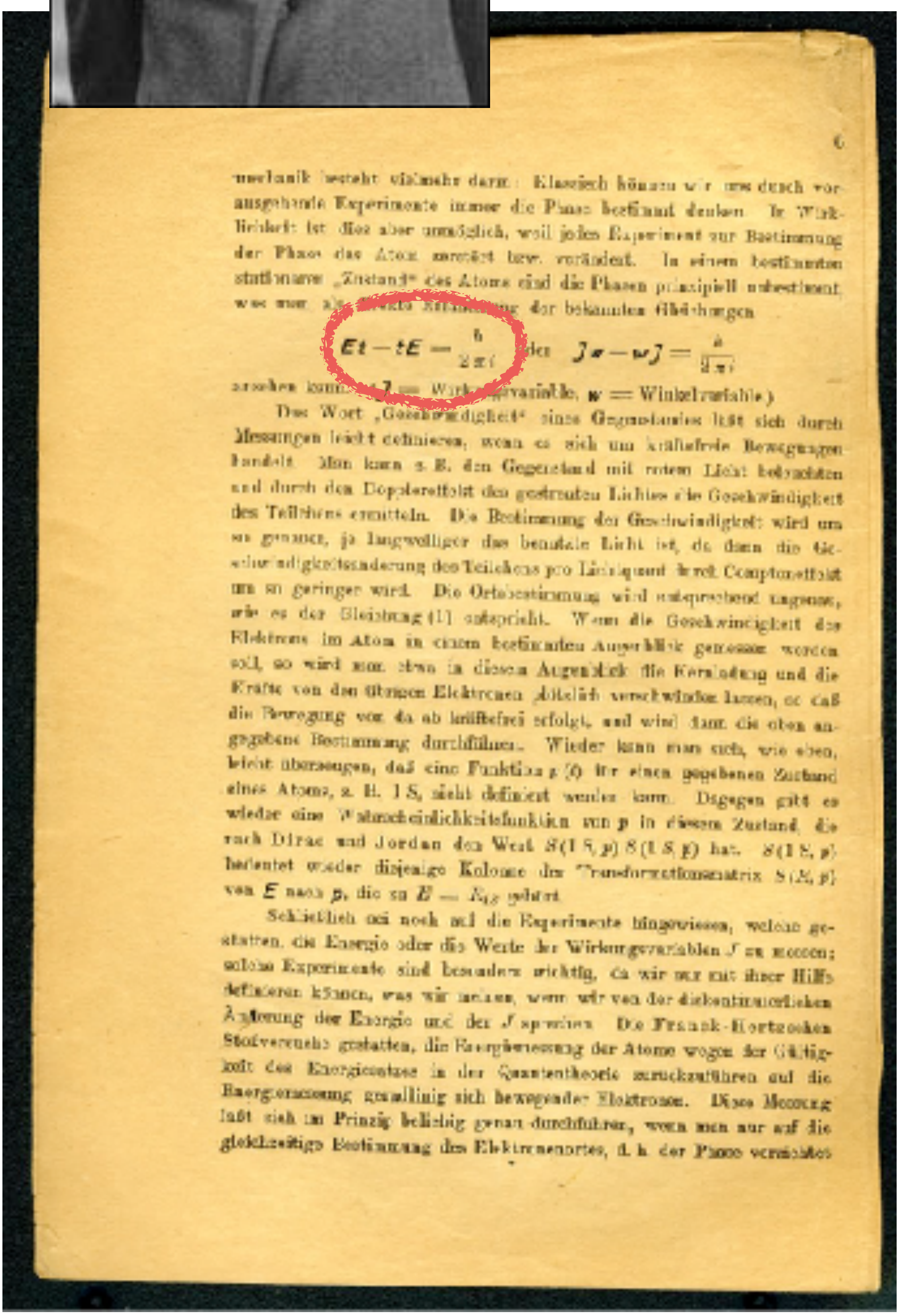


FIG. 1 (color online). Comparison between upper bound $C_Q^{\max}$ and the maximum quantum Fisher information $\mathcal{F}_Q^{\max}$ in Ref. [14] as a function of the average number of photons N. The dots stand for the values obtained in Ref. [14], the dashed line corresponds to the noiseless case ($\beta^2 = 0$), and the full lines correspond to $C_Q^{\max}$. The inset displays the two quantities up to $N = 30$, which was the range considered in Ref. [14]. From bottom to top, $\beta^2 = 5 \times 10^{-4}; 5 \times 10^{-5}; 5 \times 10^{-6}$.

Energy-time uncertainty

$$\Delta E \Delta T \geq \hbar$$



6

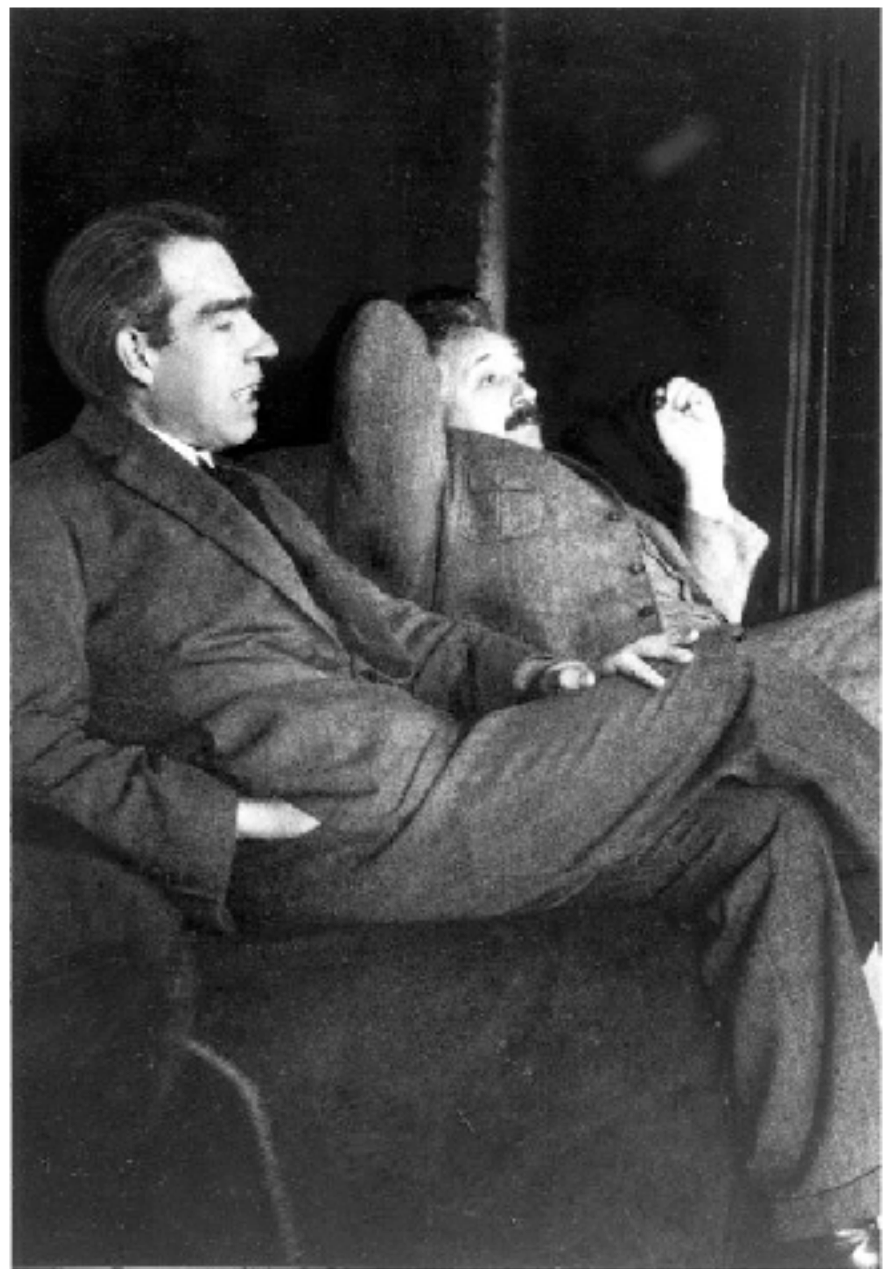
mechanik besteht vielmehr darin: Klassisch können wir uns durch vorausgehende Experimente immer die Phase bestimmt denken. In Wirklichkeit ist dies aber unmöglich, weil jedes Experiment zur Bestimmung der Phase das Atom zerstört bzw. verändert. In einem bestimmten stationären „Zustand" des Atoms sind die Phasen prinzipiell unbestimmt, was man als direkte Erläuterung der bekannten Gleichungen

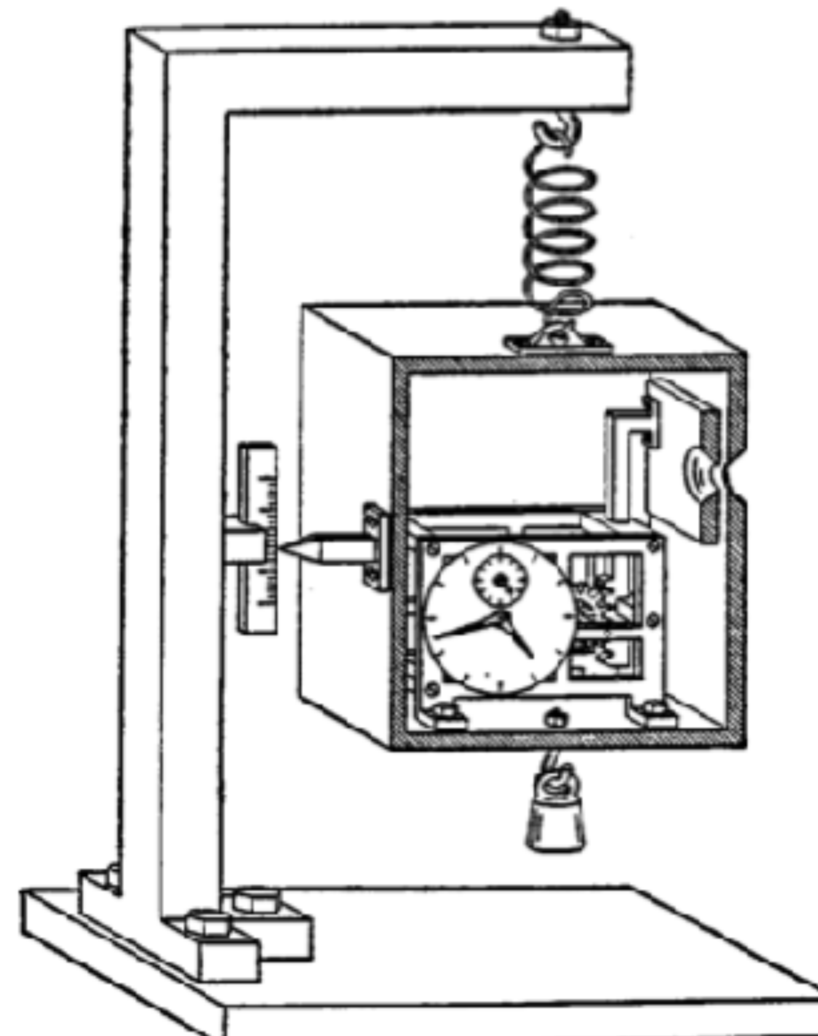
$$Et - tE = \frac{h}{2\pi i} \quad \text{oder} \quad Jw - wJ = \frac{h}{2\pi i}$$

ansehen kann. (J = Wirkungsvariable, w = Winkelvariable.)

Das Wort „Geschwindigkeit" eines Gegenstandes läßt sich durch Messungen leicht definieren, wenn es sich um kräftefreie Bewegungen handelt. Man kann z. B. den Gegenstand mit rotem Licht beleuchten und durch den Dopplereffekt des gestreuten Lichtes die Geschwindigkeit des Teilchens ermitteln. Die Bestimmung der Geschwindigkeit wird um so genauer, je langwelliger das benutzte Licht ist, da dann die Geschwindigkeitsänderung des Teilchens pro Lichtquant durch Comptoneffekt um so geringer wird. Die Ortsbestimmung wird entsprechend ungenau, wie es der Gleichung (1) entspricht. Wenn die Geschwindigkeit des Elektrons im Atom in einem bestimmten Augenblick gemessen werden soll, so wird man etwa in diesem Augenblick die Kernladung und die Kräfte von den übrigen Elektronen plötzlich verschwinden lassen, so daß die Bewegung von da ab kräftefrei erfolgt, und wird dann die oben angegebene Bestimmung durchführen. Wieder kann man sich, wie oben, leicht überzeugen, daß eine Funktion $p(t)$ für einen gegebenen Zustand eines Atoms, z. B. 1 S, nicht definiert werden kann. Dagegen gibt es wieder eine Wahrscheinlichkeitsfunktion von p in diesem Zustand, die nach Dirac und Jordan den Wert $S(1S, p)\, \bar{S}(1S, p)$ hat. $S(1S, p)$ bedeutet wieder diejenige Kolonne der Transformationsmatrix $S(E, p)$ von E nach p, die zu $E = E_{1S}$ gehört.

Schließlich sei noch auf die Experimente hingewiesen, welche gestatten, die Energie oder die Werte der Wirkungsvariablen J zu messen; solche Experimente sind besonders wichtig, da wir nur mit ihrer Hilfe definieren können, was wir meinen, wenn wir von der diskontinuierlichen Änderung der Energie und der J sprechen. Die Franck-Hertzschen Stoßversuche gestatten, die Energiemessung der Atome wegen der Gültigkeit des Energiesatzes in der Quantentheorie zurückzuführen auf die Energiemessung geradlinig sich bewegender Elektronen. Diese Messung läßt sich im Prinzip beliebig genau durchführen, wenn man nur auf die gleichzeitige Bestimmung des Elektronenortes, d. h. der Phase verzichtet





Energy-time uncertainty

THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

By L. MANDELSTAM * and Ig. TAMM

Lebedev Physical Institute, Academy of Sciences of the USSR

(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.
The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geqslant \frac{h}{2}, \qquad (1)$$

An entirely different situation is met with in the case of the relation

$$\Delta H \cdot \Delta T \sim h, \qquad (2)$$

where ΔH is the standard of energy, ΔT — a certain time interval, and the sign $\sim$ denotes that the left-hand side is at least of the order of the right-hand one.

Leonid Mandelstam

Igor Tamm

Energy-time uncertainty

Derivation of Mandelstam and Tamm is based on the relations:

$\Delta E \Delta A \geq \frac{1}{2} |\langle [H, A] \rangle|$, and $\hbar \frac{d\langle A \rangle}{dt} = i \langle [H, A] \rangle$, where A is an observable of the system ("clock observable"), not explicitly dependent on time, and H is the Hamiltonian that rules the evolution. From these two equations, we get:

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Integrating this equation with respect to time, and using that $\int_a^b |f(t)| dt \geq \left| \int_a^b f(t) dt \right|$, one gets

$$\boxed{\Delta E \Delta t \geq \frac{\hbar}{2} \left(\frac{|\langle A \rangle_{t+\Delta t} - \langle A \rangle_t|}{\overline{\Delta A}} \right)} ,$$

where $\overline{\Delta A} \equiv (1/\Delta t) \int_t^{t+\Delta t} \Delta A \, dt$ is the time average of ΔA over the integration region. We define the time interval ΔT as the shortest time for which the average value of A changes by an amount equal to its averaged standard deviation. Then $\boxed{\Delta E \Delta T \geq \hbar/2}$.

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Mandelstam and Tamm also presented a more accurate derivation, which is directly related to more modern treatments.

One starts again from

$$\Delta E \Delta A \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right|.$$

Let us choose now A to be the projection operator onto the initial state: $A = P_0 = |\psi_0\rangle\langle\psi_0|$, so that $P_0^2 = P_0$ and $\Delta P_0 = \sqrt{\langle P_0^2\rangle - \langle P_0\rangle^2} = \sqrt{\langle P_0\rangle - \langle P_0\rangle^2}$, which implies that

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Integrating this expression from 0 to τ, and using that $\int_a^b |f(t)|dt \geq \left|\int_a^b f(t)dt\right|$, one gets $\boxed{\Delta E \cdot \tau \geq \hbar \arccos\sqrt{\langle P_0\rangle_\tau}}$ where $\boxed{\langle P_0\rangle_\tau = |\psi_0|\psi_\tau|^2}$ is the fidelity between the initial and the final states. Throughout this lecture, the image of arcos is defined in $[0,\pi]$. If the final state is orthogonal to the initial one, $\langle P_0\rangle_\tau = 0$ and $\boxed{\Delta E \cdot \tau \geq h/4.}$

Energy-time uncertainty

Note that the steps leading to $\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle P_0\rangle/dt}{\sqrt{\langle P_o\rangle - \langle P_0\rangle^2}}\right|$ also hold if H depends on time. Therefore, from this equation one may extract a more general expression:

$$\boxed{\int_0^\tau \Delta E(t)\, dt \geq \hbar \arccos\sqrt{F}}$$

which is an implicit bound for the time needed to reach a fidelity $F = |\langle\psi_0|\psi_\tau\rangle|^2$ between the initial and final state.

Energy-time uncertainty

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Geometry of Quantum Evolution

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Geometric derivation. Inequality derived from the condition that actual path followed by the states should be larger than geodesic connecting the two states.

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Generalization to non-unitary processes? Life-time for decay processes? Hamiltonian should not show up!

Motivation

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5. Relation with quantum metrology

Geometrical interpretation of the quantum Fisher information

Remember that, for classical probability distributions, one had

$$\Phi_H(x,x') = \left[\sum_k \sqrt{P_k(x)P_k(x')}\right]^2 , \qquad \Phi_H(x,x') = 1 - \frac{F(x)}{4}dx^2$$

Using the expressions of the probabilities in terms of $\hat{E}_k$, the Bures fidelity between two density operators $\hat{\rho}$ and $\hat{\sigma}$ is defined as

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$$\boxed{\sqrt{\mathscr{F}_Q}/2 \to \text{ speed}}$$

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What about the distance between two states?

As seen before, the Hellinger distance between two probability distributions obeys the equation $D_H^2(x, x+dx) = ds_H^2 = [F(x)/8]dx^2$, with x a parameter.

Extension of this expression to quantum states is tricky, since the distance between $|\psi\rangle$ and $\exp(i\theta)|\psi\rangle$ or $(1+\theta)|\psi\rangle$ should be zero.

Let $|d\psi\rangle$ be the infinitesimal variation of a state $|\psi\rangle$, e.g. $|d\psi\rangle = \left(\partial|\psi\rangle/\partial\theta\right)d\theta$.
The simple metric $ds_0^2 = \langle d\psi|d\psi\rangle$ would not do, since it would yield a distance different from zero between $|\psi\rangle$ and $(1+d\theta)|\psi\rangle$: $ds_0^2 = \langle d\psi|d\psi\rangle = \langle\psi|\psi\rangle(d\theta)^2$.

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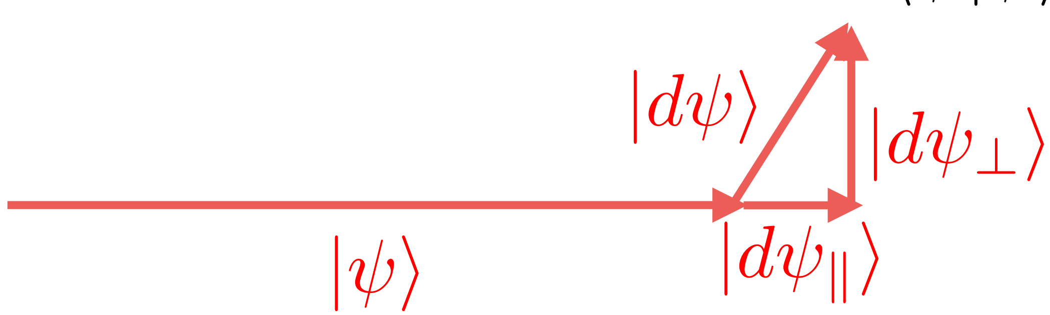

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and the norm of this angular distance as the differential form of this distance:

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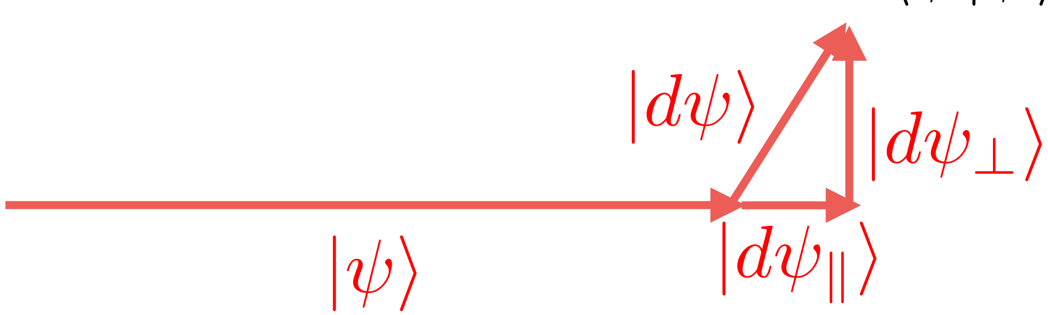

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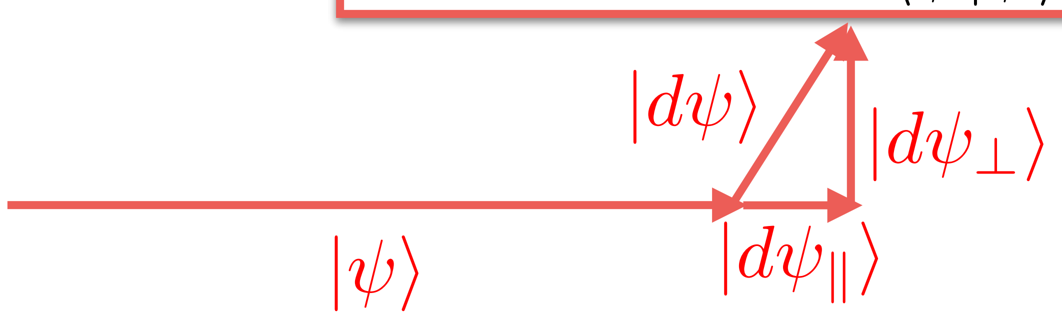

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Fubini-Study metric

Relation between distance and quantum Fisher information

$$\boxed{ds^2_{FS} = \langle d\psi_{\rm ang}|d\psi_{\rm ang}\rangle = \frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|d\psi\rangle|^2}{\langle\psi|\psi\rangle^2}}$$

Assuming that the change in $|\psi\rangle$ is due to the change in a single parameter X, one has $|d\psi\rangle = dX(d|\psi\rangle/dX)$, so that, for normalized $|\psi\rangle$,

$$\boxed{ds^2_{FS} = \left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right] dX^2}$$

Comparing this with the expression for the quantum Fisher information derived before in the first lecture:

$$\mathcal{F}_Q(X) = 4\left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right]$$

one finds that $\boxed{ds^2_{FS} = (1/4)\mathcal{F}_Q(X)dX^2}$, that is, the Fubini-Study metric is proportional to the quantum Fisher information! The larger $\mathcal{F}_Q(X)$, the more distinguishable are the states $|\psi\rangle$ and $|\psi\rangle + |d\psi\rangle$, for a given change dX of the parameter X, and therefore the better is the precision in the estimation of X.

Distance between arbitrary states

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

Integrating $ds^2_{FS} = \langle d\psi_{\rm ang}|d\psi_{\rm ang}\rangle = \frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|d\psi\rangle|^2}{\langle\psi|\psi\rangle^2}$, one gets the distance between arbitrary pure states:

$$D_{FS}(|\psi_0\rangle, |\psi_f\rangle) = \arccos\sqrt{\Phi_B(|\psi_0\rangle, |\psi_f\rangle)}$$

(maximum distance equal to $\pi/2$, for orthogonal states)

where

$$\Phi_B(|\psi_0\rangle, |\psi_f\rangle) = |\langle\psi_0|\psi_f\rangle|^2$$

is the Bures fidelity for pure states.

On a Bloch sphere, this distance would correspond to the shortest path along a great circle connecting two vectors with tips on the sphere.

For mixed states, one can show that

$$\boxed{D_B(\hat\rho_1,\hat\rho_2) == \arccos\sqrt{\Phi_B(\hat\rho_1,\hat\rho_2)}}$$

Bures angle

with

$$\boxed{\Phi_B(\hat\rho_1,\hat\rho_2) \equiv \left(\mathrm{Tr}\sqrt{\hat\rho_1^{1/2}\hat\rho_2\hat\rho_1^{1/2}}\right)^2 = |\langle\psi_1|\psi_2\rangle|^2 \ \text{(pure states)}}$$

Quantum speed limit for physical processes

M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, PRL **110**, 050402 (2013)

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Special case: Unitary evolution, time-independent Hamiltonian, orthogonal states

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Mandelstam-Tamm

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Quantum speed limit for open systems: Purification procedure

$$\mathcal{D} := \arccos \sqrt{\Phi_B\left[\hat{\rho}(0), \hat{\rho}(\tau)\right]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}\, dt$$

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$$\hat{\mathcal{H}}_{S,E}(t) := \frac{\hbar}{i} \frac{d\hat{U}^\dagger_{S,E}(t)}{dt} \hat{U}_{S,E}(t)$$

$\hat{U}_{S,E}(t)$: Evolution of purified state corresponding to $\hat{\rho}_S$

Quantum speed limit for physical processes: amplitude damping channel

The amplitude-damping channel may be described by the following equations (states without indices refer to the system — e.g. a two-level atom with $|1\rangle$ and $|0\rangle$ being the excited and ground states):

$$|0\rangle|0\rangle_E \to |0\rangle|0\rangle_E\,,$$

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This is a quite natural, physically motivated purification of the evolution of two-level atom. The unitary evolution corresponding to this map is

$$\hat{U}_{S,E}(t) = \exp[-i\Theta(t)(\hat{\sigma}_+\hat{\sigma}_-^{(E)} + \hat{\sigma}_-\hat{\sigma}_+^{(E)})] \quad \hat{\sigma}_+|0\rangle = |1\rangle \,, \quad \hat{\sigma}_-|1\rangle = |0\rangle \,, \quad \hat{\sigma}_\pm^2 = 0$$

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This implies a lower bound for the distance-dependent decay time:

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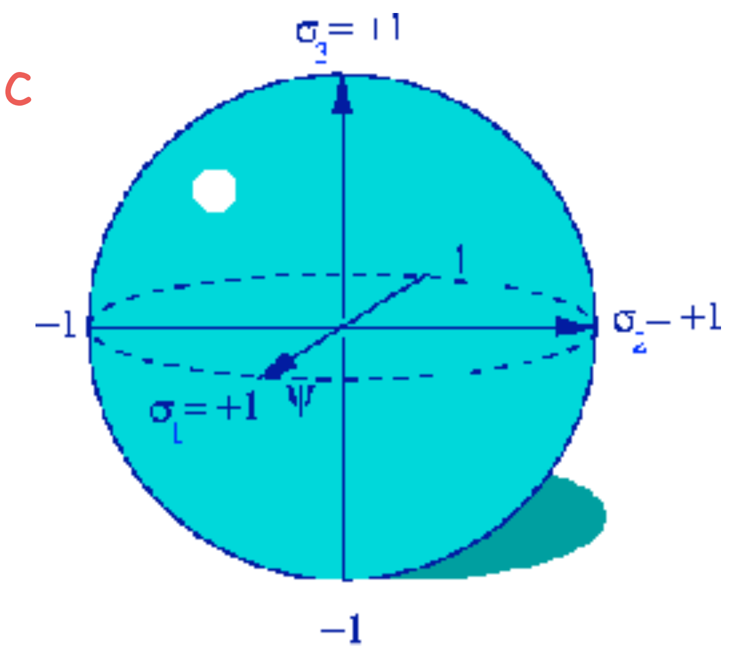

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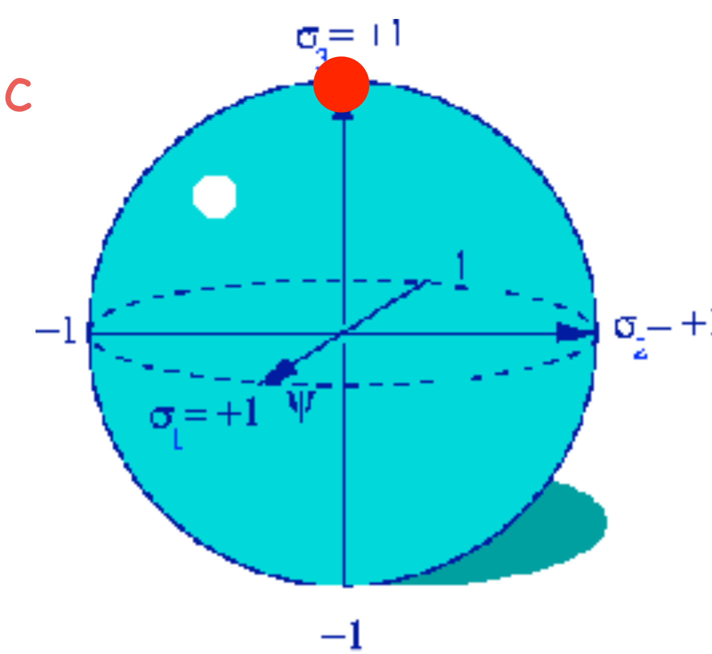

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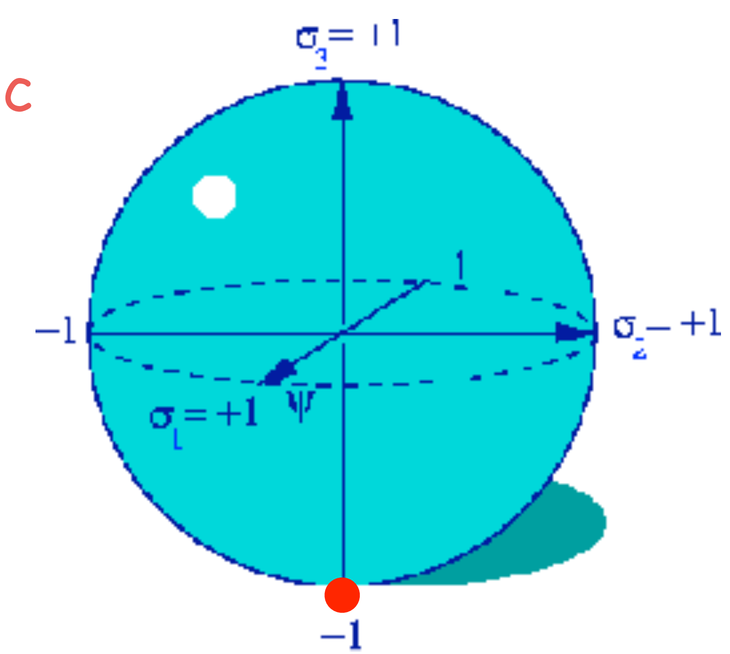

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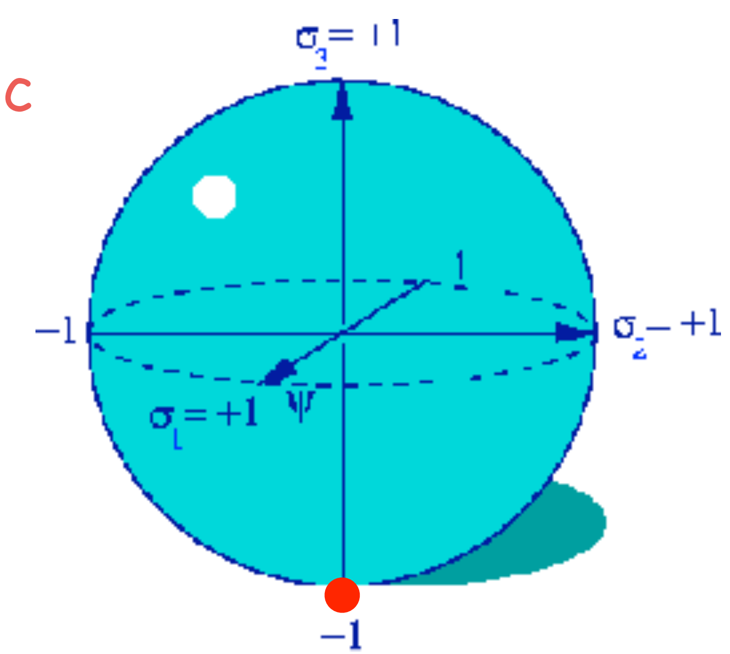

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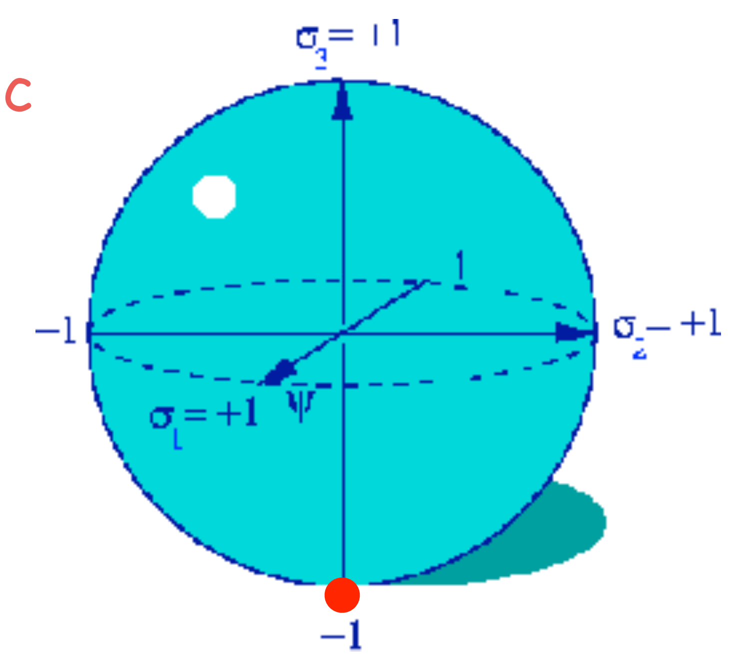

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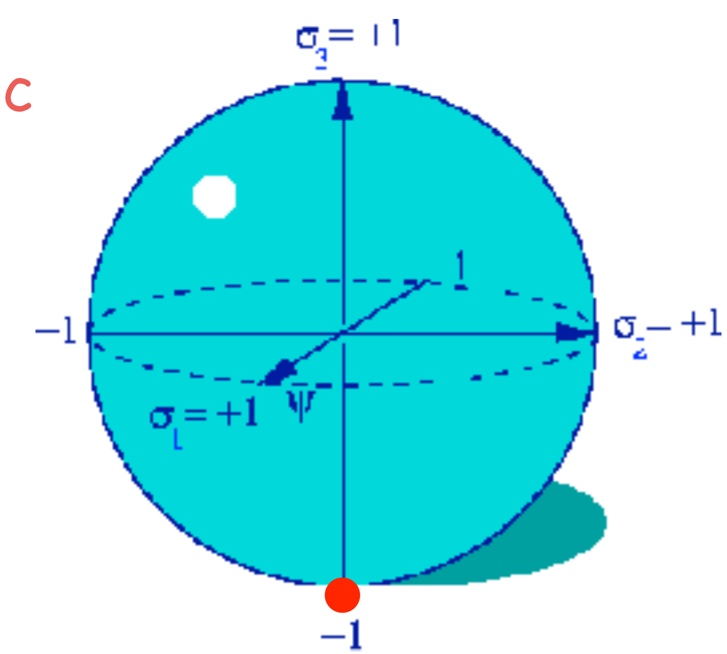

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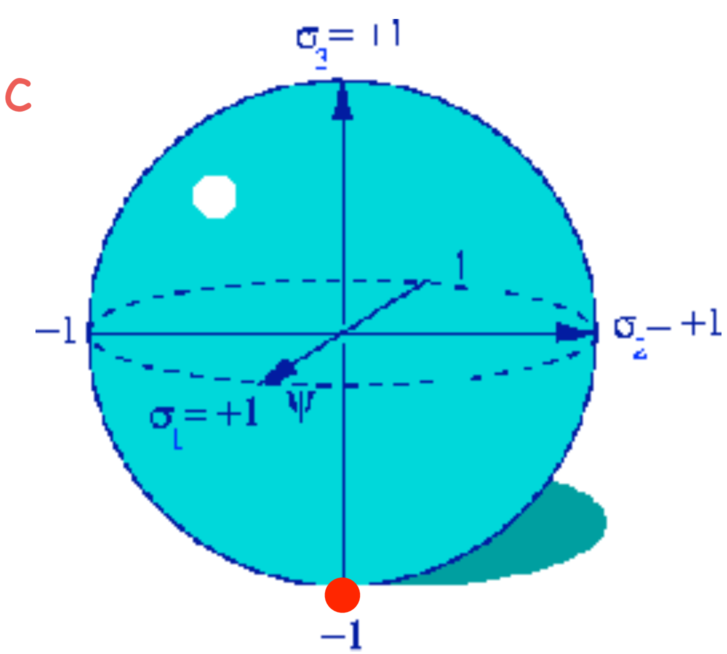

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