Cours au Collège de France - Février 2016

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

Luiz Davidovich
Instituto de Física - Universidade Federal do Rio de Janeiro

Quatrième Leçon: La relation d'incertitude énergie — temps et la limite quantique de vitesse

But de cette leçon

In this lecture, the methods developed in the previous lectures are applied to the problem of giving a precise meaning to the energy-time uncertainty relation. These methods allow the establishment of tight bounds for the speed of evolution of systems, which can be applied both to closed and open systems, thus achieving a unified treatment of the quantum speed limit. The main results are based on geometrical properties of the space of quantum states, which are introduced in this lecture, and allow a geometrical interpretation of the quantum Fisher information. Applications to atomic decay and dephasing are discussed.

Rappel sur l'Information de Fisher Quantique

In the first lecture, we defined, for a given measurement corresponding to the POVM $\{\hat{E}(\xi)\}$, the Fisher information,

$$F[X;\{\hat{E}(\xi)\}] = \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 = \int d\xi \frac{1}{p(\xi|X)}\left[\frac{\partial p(\xi|X)}{\partial X}\right]^2$$

and we have also defined the "Quantum Fisher information," which is obtained by maximizing the above expression with respect to all quantum measurements:

$$\mathcal{F}_Q(X) = \max_{\{\hat{E}(\xi)\}} F[X;\{\hat{E}(\xi)\}]$$

The lower bound for the precision in the measurement of the parameter X is then $\sqrt{\langle(\Delta X_{\rm est})^2\rangle} \geq 1/\sqrt{N\mathcal{F}_Q(X)}$, where N is the number of repetitions of the experiment.

Quantum Fisher information for pure states

We showed that the quantum Fisher information for pure states that evolve according to $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, where X is the parameter to be estimated and $\hat{U}(X)$ is a unitary operator, is

$$\boxed{\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle}$$

where

$$\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X) = -i\hat{U}^\dagger(X)\frac{d\hat{U}(X)}{dX}$$

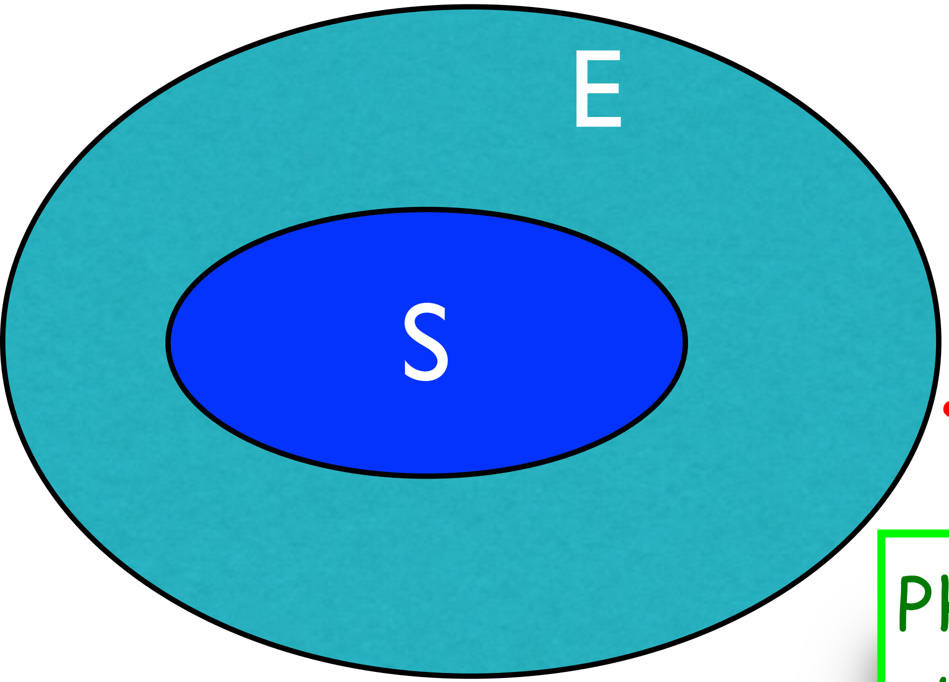
From the definition of $\hat{H}(X)$ and from the above expression, it follows that the quantum Fisher information can also be written as

$$\boxed{\mathcal{F}_Q(X) = 4\left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right]}$$

Parameter estimation in open systems: Extended space approach

B. M. Escher, R. L. Matos Filho, and L. D., **Nature Physics** 7, 406 (2011);
Braz. J. Phys. 41, 229 (2011)

Given initial state and non-unitary evolution, define in S+E



$$|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$$ (Purification)

Then

$$\mathcal{F}_Q \equiv \max_{\hat{E}_j^{(S)}\otimes\hat{1}} F\left(\hat{E}_j^{(S)}\otimes\hat{1}\right) \le \max_{\hat{E}_j^{(S,E)}} F\left(\hat{E}_j^{(S,E)}\right) \equiv \mathcal{C}_Q$$

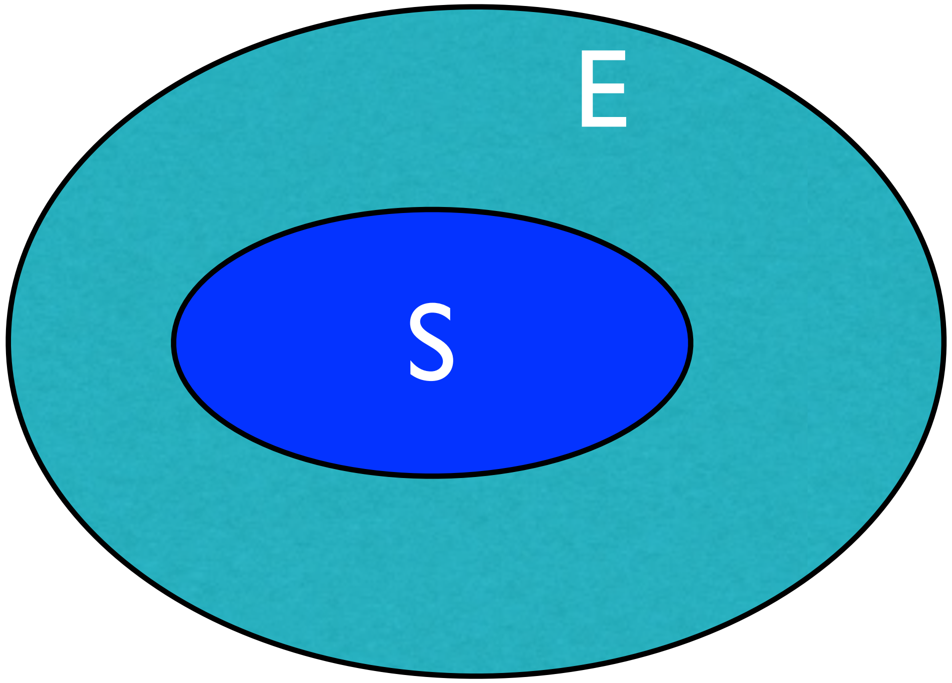
Physical meaning of this bound: information obtained about parameter when S+E is monitored

Least upper bound: Minimization over all unitary evolutions in S+E - difficult problem

Bound is attainable - there is always a purification such that $\mathcal{C}_Q = \mathcal{F}_Q$

→ Then, monitoring S+E yields same information as monitoring S

Minimization procedure



There is always an unitary operator acting only on E that connects two different purifications of $\hat{\rho}_S(x)$

Given $|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$,

$$i\frac{d|\Phi_{S,E}(x)\rangle}{dx} = \hat{H}_{S,E}(x)|\Phi_{S,E}(x)\rangle,$$

then any other purification can be written as:

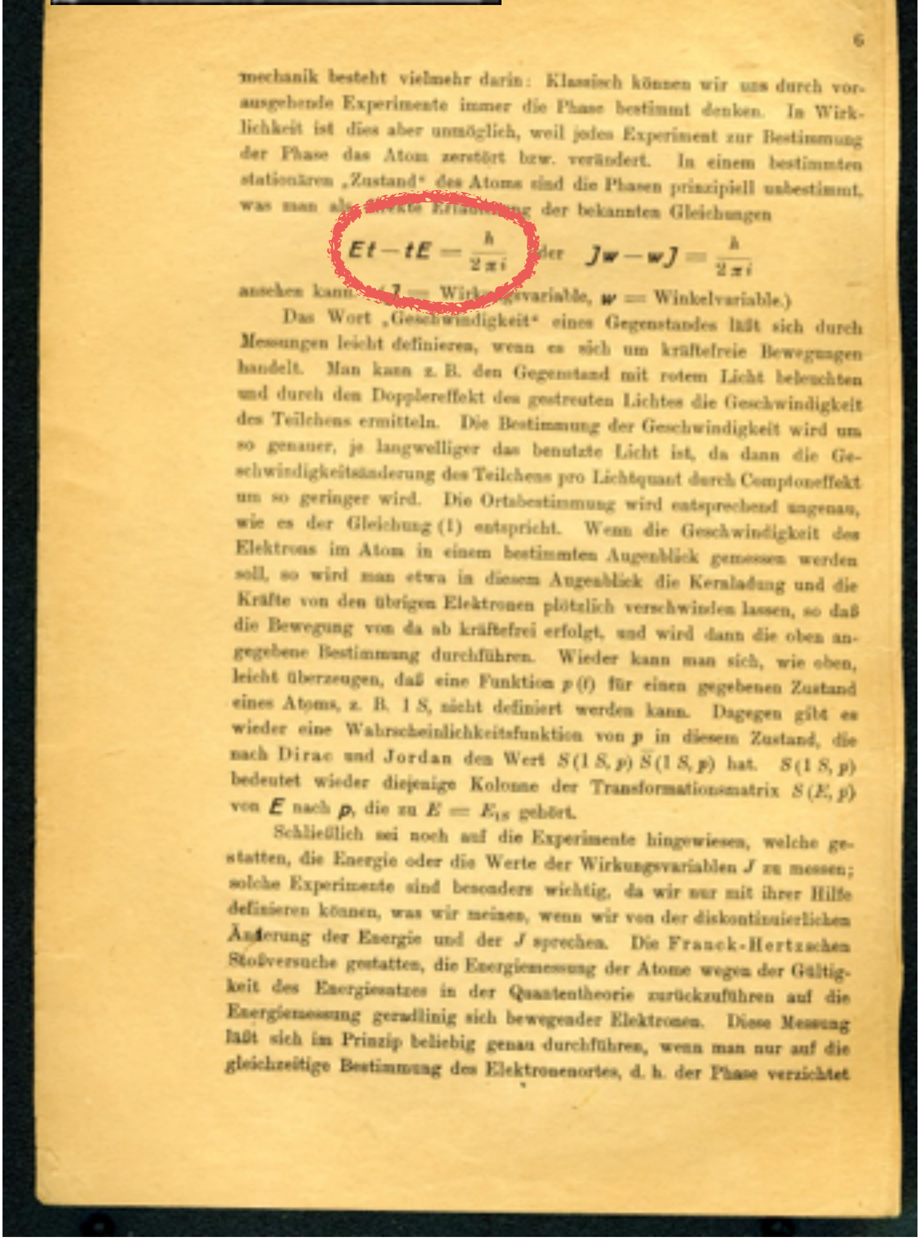
$$|\Psi_{S,E}(x)\rangle = \hat{u}_E(x)|\Phi_{S,E}(x)\rangle$$

Define $\hat{h}_E(x) = i\frac{d\hat{u}_E^\dagger(x)}{dx}\hat{u}_E(x)$

Minimize now C_Q over all Hermitian operators $\hat{h}_E(x)$ that act on E.

Energy-time uncertainty

$$\Delta E \Delta T \geq \hbar$$



6

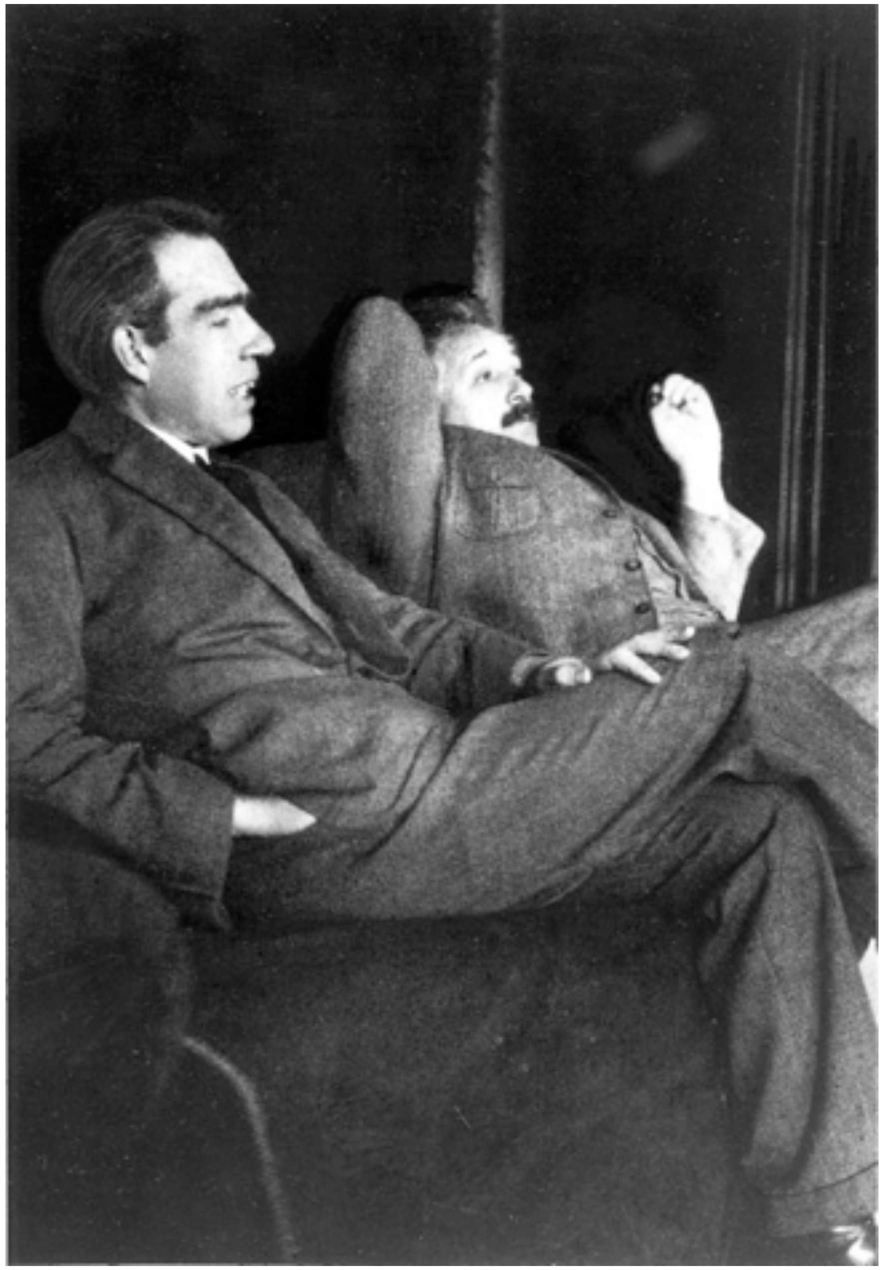
mechanik besteht vielmehr darin: Klassisch können wir uns durch vorausgehende Experimente immer die Phase bestimmt denken. In Wirklichkeit ist dies aber unmöglich, weil jedes Experiment zur Bestimmung der Phase das Atom zerstört bzw. verändert. In einem bestimmten stationären „Zustand" des Atoms sind die Phasen prinzipiell unbestimmt, was man als direkte Erläuterung der bekannten Gleichungen

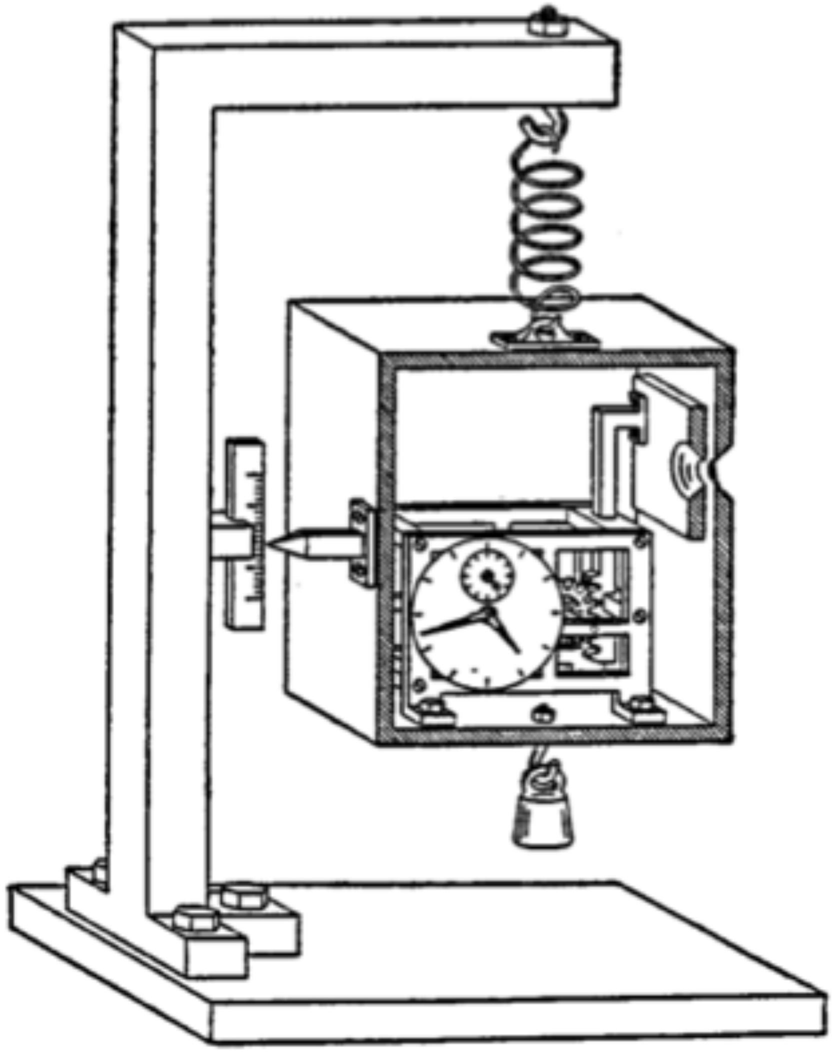
$$Et - tE = \frac{h}{2\pi i} \quad \text{oder} \quad Jw - wJ = \frac{h}{2\pi i}$$

ansehen kann. (J = Wirkungsvariable, w = Winkelvariable.)

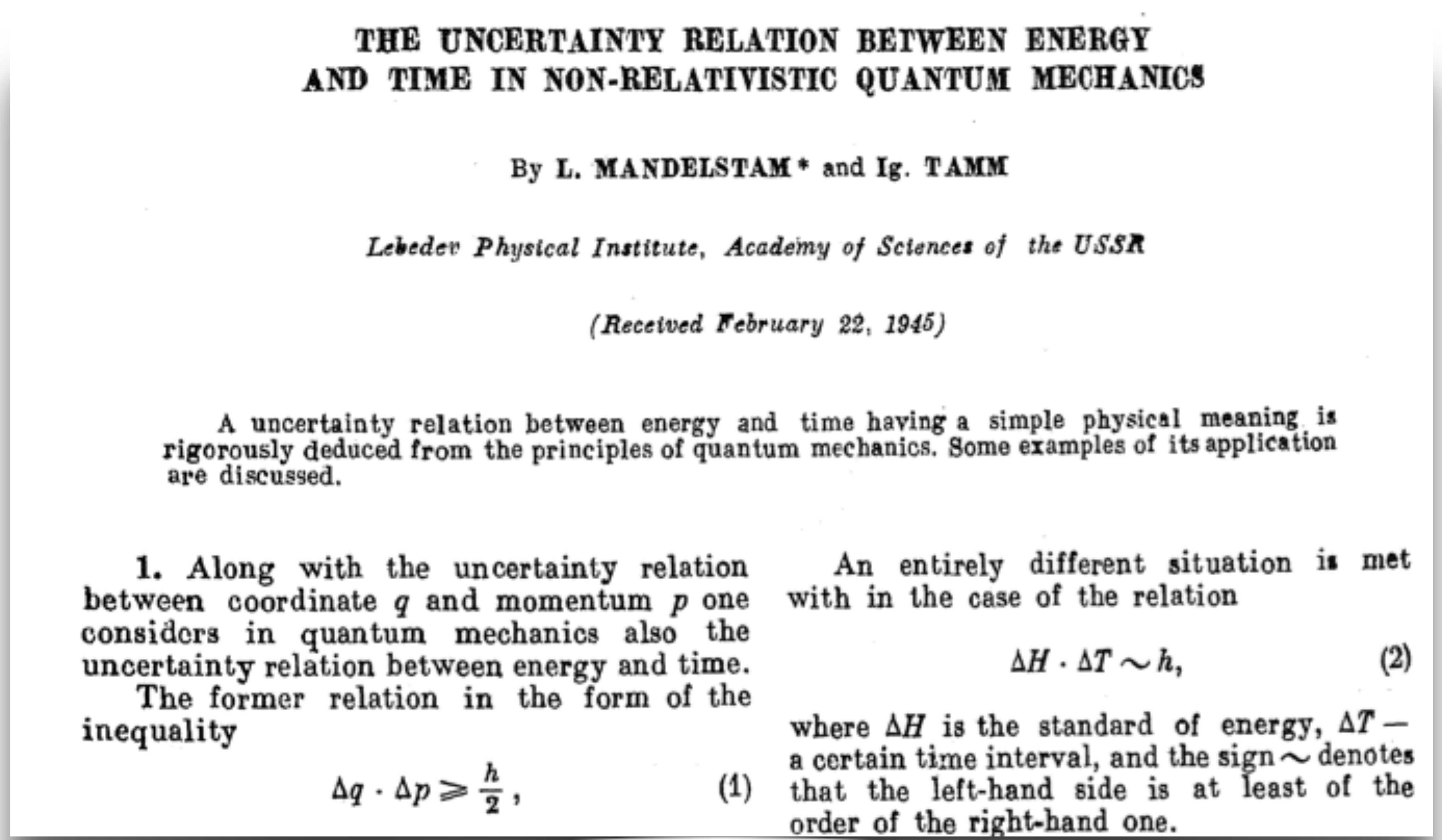
Das Wort „Geschwindigkeit" eines Gegenstandes läßt sich durch Messungen leicht definieren, wenn es sich um kräftefreie Bewegungen handelt. Man kann z. B. den Gegenstand mit rotem Licht beleuchten und durch den Dopplereffekt des gestreuten Lichtes die Geschwindigkeit des Teilchens ermitteln. Die Bestimmung der Geschwindigkeit wird um so genauer, je langwelliger das benutzte Licht ist, da dann die Geschwindigkeitsänderung des Teilchens pro Lichtquant durch Comptoneffekt um so geringer wird. Die Ortsbestimmung wird entsprechend ungenau, wie es der Gleichung (1) entspricht. Wenn die Geschwindigkeit des Elektrons im Atom in einem bestimmten Augenblick gemessen werden soll, so wird man etwa in diesem Augenblick die Kernladung und die Kräfte von den übrigen Elektronen plötzlich verschwinden lassen, so daß die Bewegung von da ab kräftefrei erfolgt, und wird dann die oben angegebene Bestimmung durchführen. Wieder kann man sich, wie oben, leicht überzeugen, daß eine Funktion $p(t)$ für einen gegebenen Zustand eines Atoms, z. B. 1 S, nicht definiert werden kann. Dagegen gibt es wieder eine Wahrscheinlichkeitsfunktion von p in diesem Zustand, die nach Dirac und Jordan den Wert $S(1\,S, p)\,\bar{S}(1\,S, p)$ hat. $S(1\,S, p)$ bedeutet wieder diejenige Kolonne der Transformationsmatrix $S(E, p)$ von E nach p, die zu $E = E_{1S}$ gehört.

Schließlich sei noch auf die Experimente hingewiesen, welche gestatten, die Energie oder die Werte der Wirkungsvariablen J zu messen; solche Experimente sind besonders wichtig, da wir nur mit ihrer Hilfe definieren können, was wir meinen, wenn wir von der diskontinuierlichen Änderung der Energie und der J sprechen. Die Franck-Hertzschen Stoßversuche gestatten, die Energiemessung der Atome wegen der Gültigkeit des Energiesatzes in der Quantentheorie zurückzuführen auf die Energiemessung geradlinig sich bewegender Elektronen. Diese Messung läßt sich im Prinzip beliebig genau durchführen, wenn man nur auf die gleichzeitige Bestimmung des Elektronenortes, d. h. der Phase verzichtet





Energy-time uncertainty



THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

By L. MANDELSTAM* and Ig. TAMM

Lebedev Physical Institute, Academy of Sciences of the USSR

(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.

The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geqslant \frac{h}{2}, \qquad (1)$$

An entirely different situation is met with in the case of the relation

$$\Delta H \cdot \Delta T \sim h, \qquad (2)$$

where ΔH is the standard of energy, ΔT — a certain time interval, and the sign $\sim$ denotes that the left-hand side is at least of the order of the right-hand one.

Leonid Mandelstam

Igor Tamm

Energy-time uncertainty

Derivation of Mandelstam and Tamm is based on the relations:

$\Delta E \Delta A \geq \frac{1}{2}|\langle[H,A]\rangle|$, and $\hbar\frac{d\langle A\rangle}{dt} = i\langle[H,A]\rangle$, where A is an observable of the system ("clock observable"), not explicitly dependent on time, and H is the Hamiltonian that rules the evolution. From these two equations, we get:

$$\Delta E \Delta A \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right| .$$

Integrating this equation with respect to time, and using that $\int_a^b |f(t)|dt \geq \left|\int_a^b f(t)dt\right|$, one gets

$$\boxed{\Delta E \Delta t \geq \frac{\hbar}{2}\left(\frac{|\langle A\rangle_{t+\Delta t} - \langle A\rangle_t|}{\overline{\Delta A}}\right)} ,$$

where $\overline{\Delta A} \equiv (1/\Delta t)\int_t^{t+\Delta t}\Delta A\, dt$ is the time average of ΔA over the integration region. We define the time interval ΔT as the shortest time for which the average value of A changes by an amount equal to its averaged standard deviation. Then $\boxed{\Delta E \Delta T \geq \hbar/2}$.

Energy-time uncertainty

Mandelstam and Tamm also presented a more accurate derivation, which is directly related to more modern treatments.

One starts again from

$$\Delta E \Delta A \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right| .$$

Let us choose now A to be the projection operator onto the initial state: $A = P_0 = |\psi_0\rangle\langle\psi_0|$, so that $P_0^2 = P_0$ and $\Delta P_0 = \sqrt{\langle P_0^2\rangle - \langle P_0\rangle^2} = \sqrt{\langle P_0\rangle - \langle P_0\rangle^2}$, which implies that

$$\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle P_0\rangle/dt}{\sqrt{\langle P_o\rangle - \langle P_0\rangle^2}}\right| .$$

Integrating this expression from 0 to τ, and using that $\int_a^b |f(t)|dt \geq \left|\int_a^b f(t)dt\right|$, one gets $\boxed{\Delta E \cdot \tau \geq \hbar \arccos \sqrt{\langle P_0\rangle_\tau}}$ where $\langle P_0\rangle_\tau = |\psi_0|\psi_\tau|^2$ is the fidelity between the initial and the final states.

Throughout this lecture, the image of arcos is defined in $[0,\pi]$. If the final state is orthogonal to the initial one, $\langle P_0\rangle_\tau = 0$ and $\boxed{\Delta E \cdot \tau \geq h/4.}$

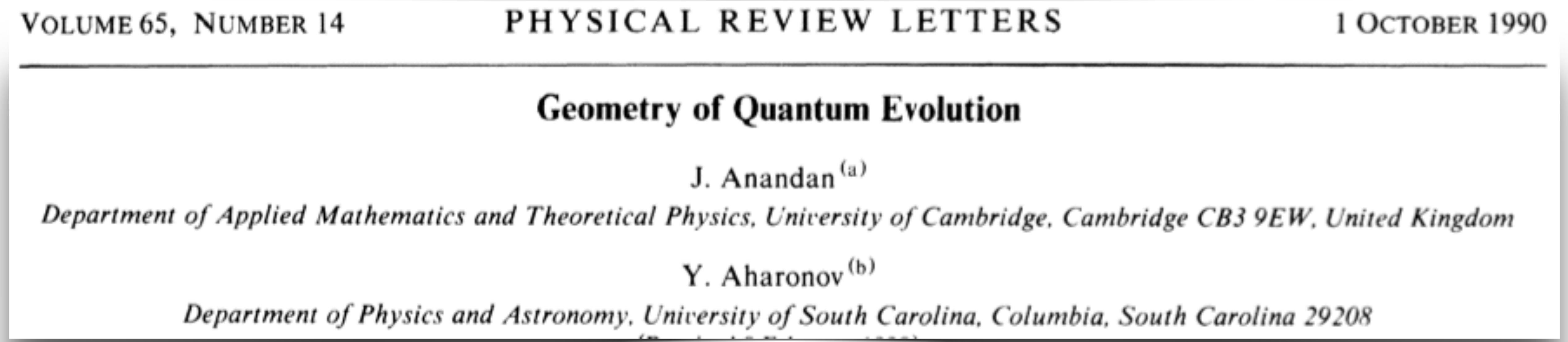
Energy-time uncertainty

Note that the steps leading to $\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle P_0\rangle/dt}{\sqrt{\langle P_o\rangle - \langle P_0\rangle^2}}\right|$ also hold if H depends on time. Therefore, from this equation one may extract a more general expression:

$$\boxed{\int_0^\tau \Delta E(t)\, dt \geq \hbar \arccos \sqrt{F}}$$

which is an implicit bound for the time needed to reach a fidelity $F = |\langle\psi_0|\psi_\tau\rangle|^2$ between the initial and final state.

Energy-time uncertainty



VOLUME 65, NUMBER 14 PHYSICAL REVIEW LETTERS 1 OCTOBER 1990

Geometry of Quantum Evolution

J. Anandan[(a)]
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

Y. Aharonov[(b)]
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208

Geometric derivation. Inequality derived from the condition that actual path followed by the states should be larger than geodesic connecting the two states.

Generalization to non-unitary processes? Life-time for decay processes? Hamiltonian should not show up!

Motivation

1. Foundations of quantum mechanics: How to interpret this relation? (Heisenberg, Einstein, Bohr, Mandelstam and Tamm, Landau and Peierls, Fock and Krylov, Aharonov and Bohm, Bhattacharyya)
2. Computation times: e.g., time taken to flip a spin — Quantum speed limit
3. Quantum-classical transition: Decoherence time
4. Control of the dynamics of a quantum system: find the fastest evolution given initial and final states and some restriction on the resources (e.g. the energy) or the general structure of the Hamiltonian.
5. Relation with quantum metrology

Some notions on the geometry of quantum states

Definition of distance between pure states

A distance is a real number that is a function of two elements of a set, say x and y. The three defining properties of a distance are:

(i) $D(x,y) \geq 0 \quad \text{and} \quad D(x,y) = 0 \Leftrightarrow x = y$

(ii) $D(x,y) = D(y,x)$

(iii) $D(x,z) \leq D(x,y) + D(y,x)$ (triangle inequality)

How to define a distance between quantum states? Since two vectors of Hilbert space that differ by a constant actually correspond to the same quantum state, one would like to have a definition of distance that should be zero between states that differ by a constant, like $|\psi\rangle$ and $\lambda|\psi\rangle$. This means that the distance will be defined in a **projective Hilbert space**. A projective space is obtained from a vector space by identifying vectors that differ by a nonzero factor.

Some notions on the geometry of quantum states

In order to define a distance, one needs a metric, in analogy to the Riemannian metric in Euclidian space:

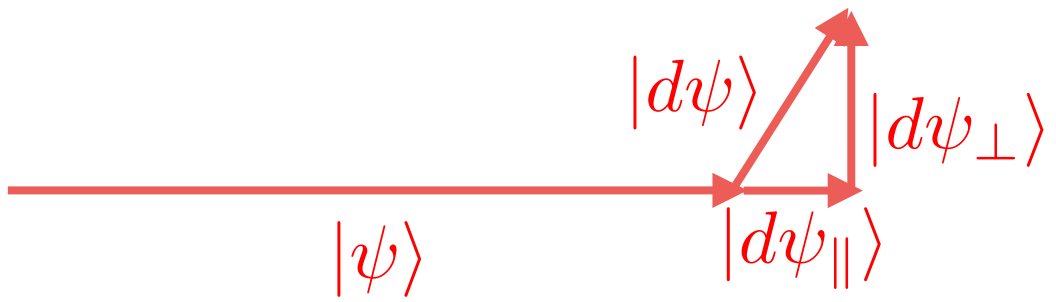
$$ds^2 = dx^2 + dy^2 + dz^2$$

Let $|d\psi\rangle$ be an infinitesimal variation of $|\psi\rangle$, due to the variation of some parameter X on which the state depends, so that $|d\psi\rangle = dX(d|\psi\rangle/dX)$. Then, one possibility would be to define the metric $ds_0^2 = \langle d\psi|d\psi\rangle$. But this definition would lead to a distance different from zero between $|\psi\rangle$ and $\exp(iX)|\psi\rangle$ or $(1+X)|\psi\rangle$, which correspond in fact to the same state.

Distance between pure states (1)

We need therefore a differential form that does not distinguish parallel vectors (this means that we are looking for a metric in **projective space**, which includes non-normalized states). In order to do this, one starts with

$$|d\psi_\perp\rangle := |d\psi\rangle - \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}|d\psi\rangle$$



(Note that $|d\psi\rangle = dz|\psi\rangle \Rightarrow |d\psi_\perp\rangle = 0$).

which defines the component of $|d\psi\rangle$ orthogonal to $|\psi\rangle$. From this expression, one defines the "angular distance" (or "projective distance")

$$|d\psi_{\text{ang}}\rangle := \frac{|d\psi_\perp\rangle}{\sqrt{\langle\psi|\psi\rangle}} = \frac{|d\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}} - \frac{\langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle^{3/2}}|\psi\rangle$$

(Measure of changes in projective space)

The norm of this angular distance yields the differential form of the distance:

$$\boxed{ds^2_{FS} = \langle d\psi_{\text{ang}}|d\psi_{\text{ang}}\rangle = \frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|d\psi\rangle|^2}{\langle\psi|\psi\rangle^2}}$$

which is the Fubini-Study metric (invariant under any unitary U applied to both $|\psi\rangle$ and $|\psi\rangle + |d\psi\rangle$), which does not have the inconvenient features mentioned before. From this expression, the finite distance between two states can be obtained.

Distance between pure states (2)

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

The finite distance between two states is obtained by integrating

$$ds^2_{FS} = \langle d\psi_{\text{ang}} | d\psi_{\text{ang}} \rangle = \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{|\langle \psi | d\psi \rangle|^2}{\langle \psi | \psi \rangle^2}$$

along the shortest path (geodesic) in state space.

It can be shown that this geodesic lies entirely in a two-dimensional subspace of the vector space, spanned by the initial and final states. This can be motivated by the analogy with a unit sphere, for which the geodesics — the great circles — lie in a plane containing the origin. This implies that the geodesic can be expressed as a parametrized superposition of the initial and final states.

Let $|\psi_0\rangle$ and $|\psi_f\rangle$ be the initial and final states, and let $|\psi_1\rangle$ be a state orthogonal to $|\psi_0\rangle$ and belonging to the two-dimensional space spanned by $|\psi_0\rangle$ and $|\psi_f\rangle$. The state along the geodesics can be written as $|\psi(s)\rangle = f(s)|\psi_0\rangle + g(s)|\psi_1\rangle$, where s is a real parameter, f(s) and g(s) are complex functions of s, and the states are not necessarily normalized (rays in Hilbert space). Inserting this into ds^2_{FS}, integrating, and finding the path [that is, the functions f(s) and g(s)] that minimizes the length, one finds the finite distance D_{FS} between $|\psi_0\rangle$ and $|\psi_f\rangle$.

Distance between pure states (3)

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

$$\boxed{D_{FS}(|\psi_0\rangle, |\psi_f\rangle = \arccos\left(\frac{|\langle\psi_0|\psi_f\rangle|}{\sqrt{\langle\psi_0|\psi_0\rangle}\sqrt{\langle\psi_f|\psi_f\rangle}}\right).}$$

The maximum value of this distance is $\pi/2$, corresponding to orthogonal states.

The argument of the arc cosine above is the square root of the fidelity between the two states, so we can also write

$$D_{FS}(|\psi_0\rangle,\, |\psi_f\rangle) = \arccos\sqrt{F(|\psi_0\rangle,\, |\psi_f\rangle)}$$

On a Bloch sphere, this distance would correspond to the shortest path along a great circle connecting two vectors with tips on the sphere.

With these geometrical notions, one is able now to derive the Mandelstam-Tamm bound geometrically, in a very simple way. Before doing that, we compare the above distance with an alternative expression.

Another possible distance

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

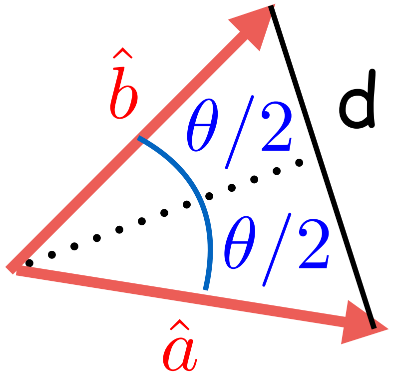
A distance in quantum state space that also satisfies all the three properties above is (this was also defined by Bures):

$$\boxed{D(|\psi_0\rangle, |\psi_f\rangle) = \sqrt{2}\sqrt{1 - \sqrt{F(|\psi_0\rangle, |\psi_f\rangle)}}}$$

where, as before,

$$F(|\psi_0\rangle, |\psi_f\rangle) = \frac{|\langle\psi_0|\psi_f\rangle|^2}{\langle\psi_0|\psi_0\rangle\langle\psi_f|\psi_f\rangle}$$

This is analogous to the distance between two unit vectors, as shown in the figure below.



$$d = 2\sin(\theta/2) = 2\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{2}\sqrt{1-\cos\theta} = \sqrt{2}\sqrt{1-\hat{a}\cdot\hat{b}}$$

Another possible distance (2)

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

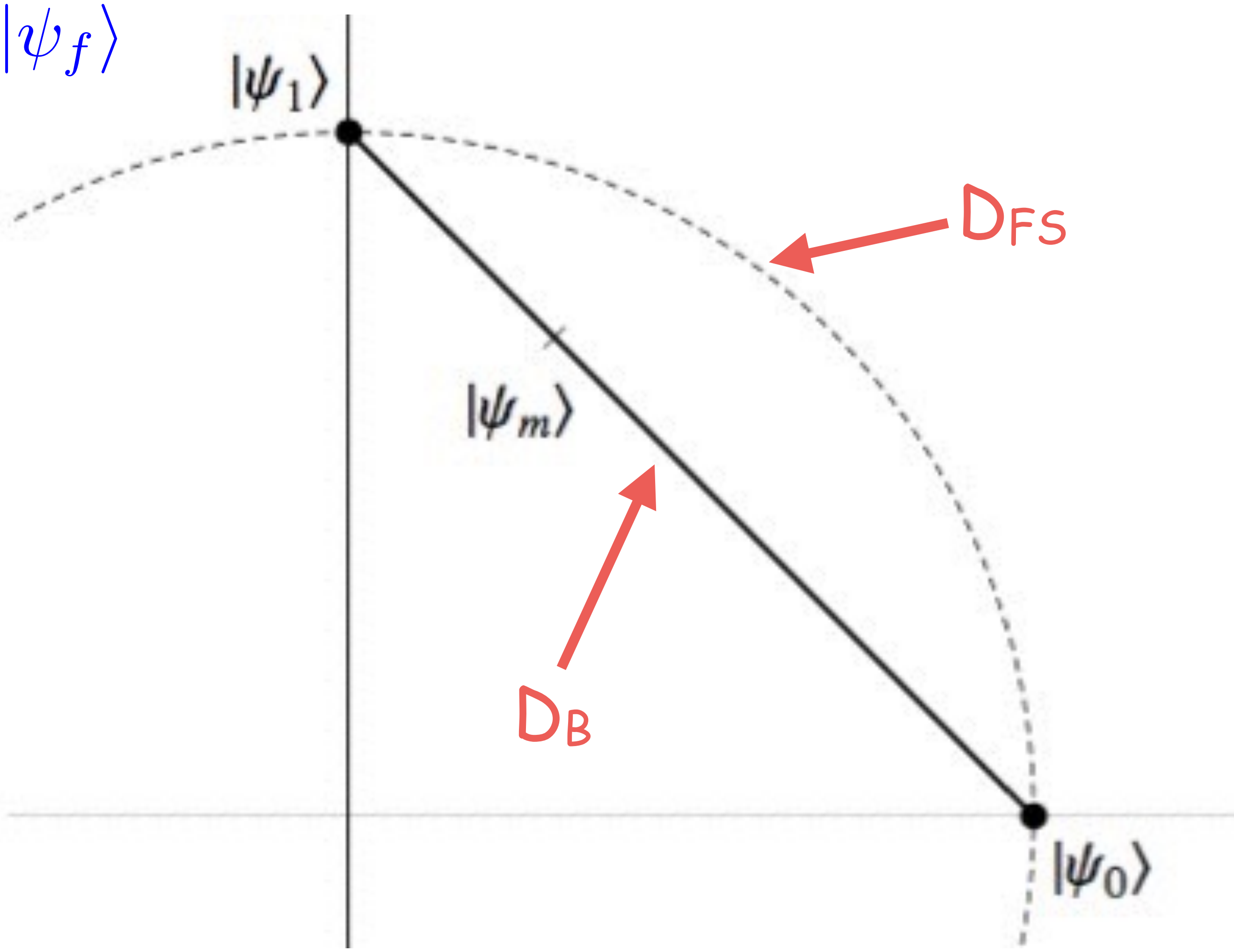
A distance in quantum state space that also satisfies all the three properties above is (this was also defined by Bures):

$$\boxed{D(|\psi_0\rangle, |\psi_f\rangle) = \sqrt{2}\sqrt{1 - \sqrt{F(|\psi_0\rangle, |\psi_f\rangle)}}}$$

where, as before,

$$F(|\psi_0\rangle, |\psi_f\rangle) = \frac{|\langle\psi_0|\psi_f\rangle|^2}{\langle\psi_0|\psi_0\rangle\langle\psi_f|\psi_f\rangle}$$

This distance cannot be obtained however as the shortest path along elements of the projective space, since it involves a path that contains necessarily unnormalized states, like $|\psi_m\rangle$. It cannot be obtained from ds^2_{FS}, which is independent of normalization.



Geometric derivation of the Mandelstam-Tamm bound

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

Let us calculate the differential form of the Fubini-Study metric when the variation of $|\psi\rangle$ is due to an evolution operator corresponding to the Hamiltonian H. Then $|d\psi(t)\rangle = |\psi(t+dt)\rangle - |\psi(t)\rangle = (H/i\hbar)|\psi(t)\rangle dt,$ where the parameter s is now the time. Replacing this into the expression for ds^2_{FS}:

$$ds^2_{FS} = \frac{1}{\hbar^2}\left[\frac{\langle\psi|H^2|\psi\rangle}{\langle\psi|\psi\rangle} - \left(\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}\right)^2\right] dt^2 = \frac{(\Delta E)^2\, dt^2}{\hbar^2}$$

Integrating ds_{FS} along the path followed by the state, one obtains the length of this path:

$$\ell_{FS} = \int ds_{FS} = \int_0^\tau \frac{\Delta E(t)}{\hbar} dt.$$

Length of actual path followed by the state, dictated by H.

The Mandelstam-Tamm bound is obtained by remarking that this distance cannot be smaller than the length of the geodesic connecting the two states:

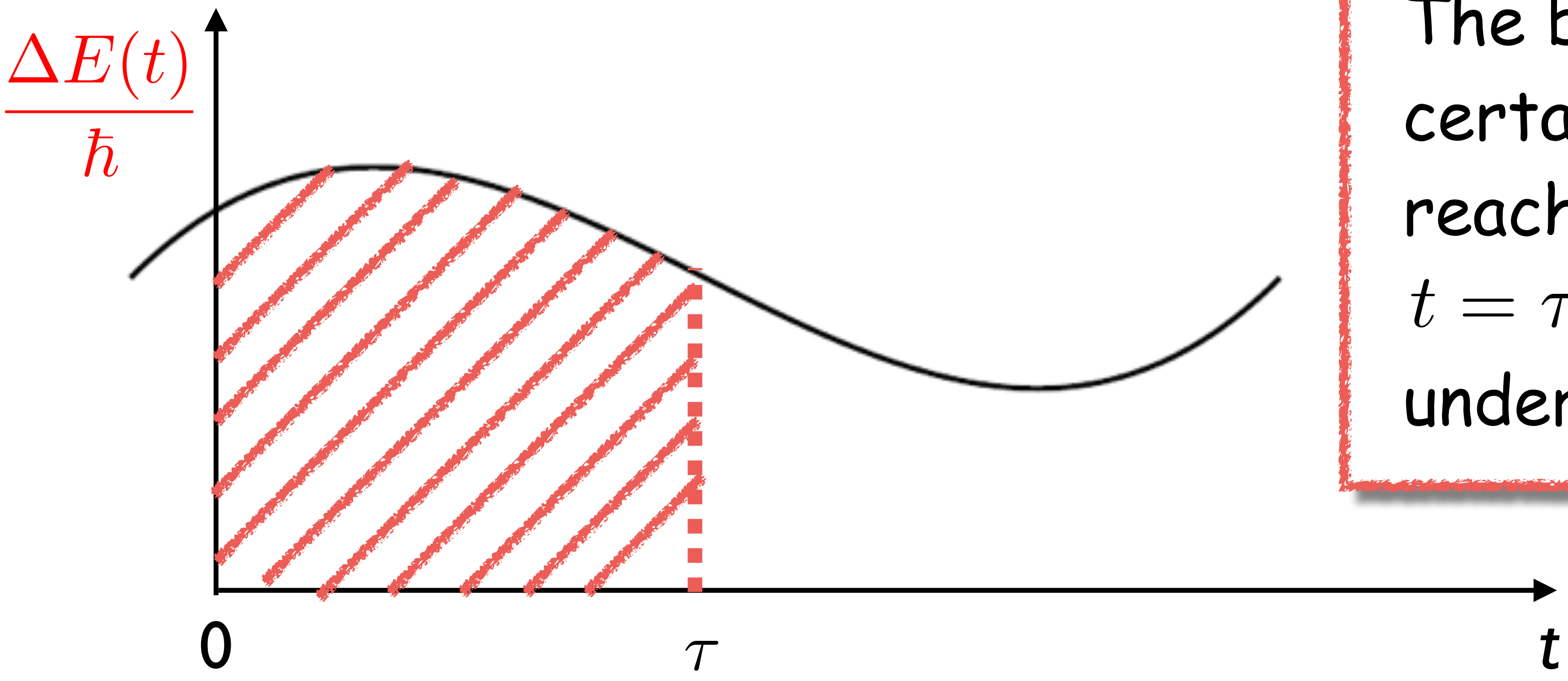
$$\boxed{D_{FS}(|\psi_0\rangle, |\psi_f\rangle) = \arccos\sqrt{F(|\psi_0\rangle, |\psi_f\rangle)} \leq \int_0^t \frac{\Delta E(t)}{\hbar} dt}$$

Geometric derivation of the Mandelstam-Tamm bound (2)

See Marcio Taddei, Ph. D. thesis, arxiv.org/pdf/1407.4343

$$D_{FS}(|\psi_0\rangle, |\psi_f\rangle) = \arccos\sqrt{F(|\psi_0\rangle, |\psi_f\rangle)} \leq \int_0^t \frac{\Delta E(t)}{\hbar}dt$$

This expression can be interpreted in the following way: it yields the minimal time necessary for the distance between states $|\psi_0\rangle$ and $|\psi_f\rangle$ to reach a chosen value (or, equivalently, for the fidelity between these states to reach a chosen value).



The bound on time for a certain distance D₁ to be reached is given by the value $t = \tau$ such that the area under the graph equals D₁.

Geometric interpretation of the quantum Fisher information

$$\boxed{ds^2_{FS} = \langle d\psi_{\rm ang}|d\psi_{\rm ang}\rangle = \frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|d\psi\rangle|^2}{\langle\psi|\psi\rangle^2}}$$

Assuming that the change in $|\psi\rangle$ is due to the change in a single parameter X, one has $|d\psi\rangle = dX(d|\psi\rangle/dX)$, so that, for normalized $|\psi\rangle$,

$$ds^2_{FS} = \frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2 dX^2$$

Comparing this with the expression for the quantum Fisher information derived in the second lecture:

$$\mathcal{F}_Q(X) = 4\left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right]$$

one finds that $\boxed{ds^2_{FS} = (1/4)\mathcal{F}_Q(X)dX^2}$ that is, the Fubini-Study metric is proportional to the quantum Fisher information! The larger $\mathcal{F}_Q(X)$, the more distinguishable are the states $|\psi\rangle$ and $|\psi\rangle + |d\psi\rangle$, for a given change dX of the parameter X, and therefore the better is the precision in the estimation of X.

Distance for mixed states

As shown before, the distance between two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ is $D_{FS}(|\psi_0\rangle,\, |\psi_f\rangle) = \arccos\sqrt{F(|\psi_0\rangle,\, |\psi_f\rangle)}$, where, for normalized states, the fidelity is $F(|\psi_1\rangle,\, |\psi_2\rangle) = |\langle\psi_1|\psi_2\rangle|^2$

The corresponding expression for mixed states is obtained from the Bures metric, which is a generalization of the Fubini-Study metric:

$$\boxed{D_B(\hat{\rho}_1,\hat{\rho}_2) == \arccos\sqrt{\Phi_B(\hat{\rho}_1,\hat{\rho}_2)}}$$

Bures angle

where $\Phi_B(\rho_1,\rho_2)$ is the Bures fidelity, given by

$$\boxed{\Phi_B(\hat{\rho}_1,\hat{\rho}_2) \equiv \left(\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}\right)^2 = |\langle\psi_1|\psi_2\rangle|^2 \ \text{(pure states)}}$$

Uhlmann demonstrated that the Bures fidelity can be defined in terms of purifications. Let $|\Psi_1\rangle$ and $|\Psi_2\rangle$ be purifications of ρ_1 and ρ_2, respectively. Then $\Phi_B(\rho_1,\rho_2) = \max_{|\Psi_2\rangle} |\langle\Psi_1|\Psi_2\rangle|^2$, where the maximum is taken over all possible purifications of ρ_2. It is sufficient to consider "environments" with the same dimension as the system S. This motivates the definition of Bures fidelity. We demonstrate now the above expression for $D_B(\rho_1,\rho_2)$.

Distance for mixed states (2)

The differential form of the distance between two neighboring states ρ and $\rho + d\rho$ is defined as the minimal Fubini-Study differential of the respective purifications $|\Psi\rangle$ and $|\Psi\rangle + |d\Psi\rangle$:

$$ds^2_B\Big|_{\rho,\rho+d\rho} = \min_{\text{purif}} ds^2_{FS}\Bigg|_{|\Psi\rangle,|\Psi\rangle+|d\Psi\rangle} . \qquad \text{(Bures)}$$

The corresponding length is

$$\ell_B = \int_{\text{path}} ds_B = \int_{\text{path}} \min_{\text{purif}} ds_{FS} = \min_{\text{purif}} \int_{\text{path}} ds_{FS} = \min_{\text{purif}} \ell_{FS},$$

where the minimization is performed over all purifications of each state in the path.

The distance between ρ_1 and ρ_2 is now defined as the length of the shortest path between these states:

$$D_B(\rho_1, \rho_2) = \min_{\text{path}} \ell_B = \min_{\text{path}} \left\{ \min_{\text{purif}} \ell_{FS} \right\}$$

The order of the minimizations can be inverted.

Distance for mixed states (3)

Therefore:

$$D_B(\rho_1,\rho_2) = \min_{\text{purif}} \left\{ \min_{\text{path}} \ell_{FS} \right\} = \min_{\text{purif}} D_{FS}(|\Psi_1\rangle, |\Psi_2\rangle) = \min_{\text{purif}} \arccos \sqrt{F(|\Psi_1\rangle, |\Psi_2\rangle)}$$

Since D_{FS} is a decreasing function of the fidelity F, one has

$$\boxed{D_B(\rho_1,\rho_2) = \arccos \sqrt{\max_{\text{purif}} F(|\Psi_1\rangle, |\Psi_2\rangle)} = \arccos \sqrt{\Phi_B(\rho_1,\rho_2)},}$$

which demonstrates the generalization of D_{FS} for mixed states — the **Bures angle**.

Let now $\rho_1 = \rho(X)$, $\rho_2 = \rho(X+dX)$, where X is a parameter, and let us expand D_B as function of dX. It follows then that

$$\Phi_B[\rho(X), \rho(X+dX)] = 1 - \frac{\mathcal{F}_Q(X)}{4} dX^2 + \mathcal{O}(dX^4)$$

and, using that $\arccos\sqrt{1-x} = \sqrt{x} + \mathcal{O}(x^{3/2})$,

$$\boxed{D_B[\rho(X), \rho(X+dX)] = ds_B = (1/2)\sqrt{\mathcal{F}_Q(X)}dX}$$

implying that $(1/2)\sqrt{\mathcal{F}_Q(X)}$ is the speed of change of the distance between the two states.

Quantum speed limit for physical processes

M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, PRL **110**, 050402 (2013)

The previous results imply an extension to open systems of the Mandelstam-Tamm relation:

$$\boxed{\arccos\sqrt{\Phi_B[\hat{\rho}(0),\hat{\rho}(\tau)]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)}/2dt}$$

$\Longrightarrow$ Lower bound for time needed to reach fidelity $\Phi_B\left[\hat{\rho}(0),\hat{\rho}(\tau)\right]$ between initial and final states

Bures length of geodesic

Bures length of actual path followed by state of the system

Special case: Unitary evolution, time-independent Hamiltonian, orthogonal states

Mandelstam-Tamm

$$\Phi_B\left[\hat{\rho}(0),\hat{\rho}(\tau)\right]=0,\quad \mathcal{F}_Q(t)=4\langle(\Delta H)^2\rangle/\hbar^2 \Rightarrow \boxed{\tau\sqrt{\langle(\Delta H)^2\rangle}\geq h/4}$$

Quantum speed limit for open systems: Purification procedure

$$\mathcal{D} := \arccos\sqrt{\Phi_B\left[\hat{\rho}(0),\hat{\rho}(\tau)\right]} \le \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}\, dt$$

$$\Downarrow$$

Problem: No analytical expression for $\mathscr{F}_Q$

$\Rightarrow$ Purification!

$$\mathcal{D} \le \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}\, dt = \int_0^\tau \sqrt{\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle}/\hbar\, dt.$$

$$\hat{\mathcal{H}}_{S,E}(t) := \frac{\hbar}{i}\frac{d\hat{U}^\dagger_{S,E}(t)}{dt}\hat{U}_{S,E}(t)$$

$\hat{U}_{S,E}(t)$: Evolution of purified state corresponding to $\hat{\rho}_S$

Quantum speed limit for physical processes: amplitude damping channel

As seen in Lecture 3, the amplitude-damping channel may be described by the following equations (states without indices refer to the system — e.g. a two-level atom with $|1\rangle$ and $|0\rangle$ being the excited and ground states):

$$|0\rangle|0\rangle_E \to |0\rangle|0\rangle_E \,,$$

$$|1\rangle|0\rangle_E \to \sqrt{P(t)}|1\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E \qquad P(t) = \exp(-\gamma t)$$

This is a quite natural, physically motivated purification of the evolution of two-level atom. The unitary evolution corresponding to this map is

$$\hat{U}_{S,E}(t) = \exp[-i\Theta(t)(\hat{\sigma}_+\hat{\sigma}_-^{(E)} + \hat{\sigma}_-\hat{\sigma}_+^{(E)})] \quad \hat{\sigma}_+|0\rangle = |1\rangle\,, \quad \hat{\sigma}_-|1\rangle = |0\rangle\,, \quad \hat{\sigma}_\pm^2 = 0$$

$$\hat{\sigma}_+\hat{\sigma}_- = |1\rangle\langle 1|$$

with $\Theta(t) = \arccos\sqrt{P(t)}$.

From this and $\mathcal{D} \leq \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}\; dt = \int_0^\tau \sqrt{\langle \Delta\hat{\mathcal{H}}_{S,E}^2(t)\rangle}/\hbar\, dt.$

one gets: $\mathcal{D} \leq \sqrt{\langle\hat{\sigma}_+\hat{\sigma}_-\rangle}\arccos[\exp(-\gamma t/2)]$

Initial population of excited state

Quantum speed limit for physical processes: amplitude damping channel (2)

This implies a lower bound for the distance-dependent decay time:

$$\mathcal{D} \leq \sqrt{\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle} \arccos[\exp(-\gamma\tau/2)] \Rightarrow \boxed{\gamma\tau \geq 2 \ln \sec(\mathcal{D}/\sqrt{\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle})}$$

Initial population of excited state

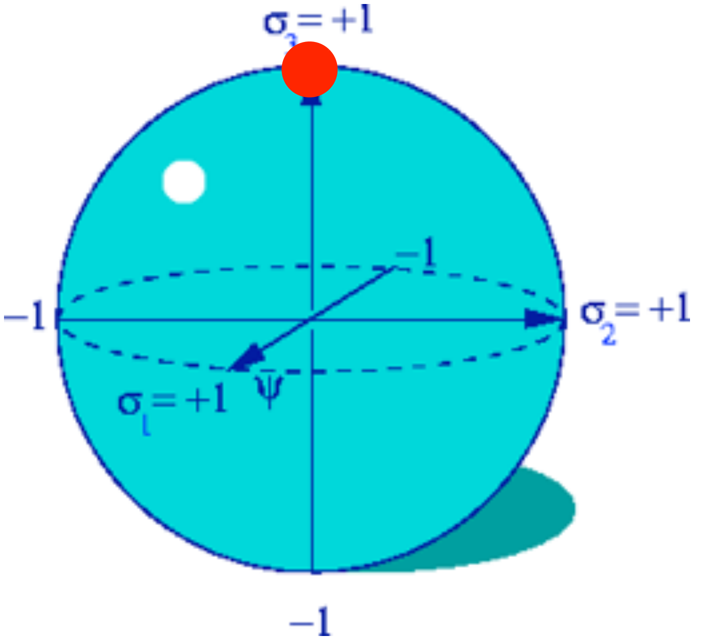
Bound is saturated if $\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle = 0 \text{ or } 1$

$$\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle = 1 \Rightarrow \; |1\rangle\langle 1| \to P(t)|1\rangle\langle 1| + [1 - P(t)]|0\rangle\langle 0|$$

$$\Rightarrow \Phi = \sqrt{P(\tau)} \Rightarrow \mathcal{D} = \arccos[\exp(-\gamma\tau/2)]$$

Interpretation:

If initial state is the excited state, then evolution is along a geodesic



Time for getting at the origin:

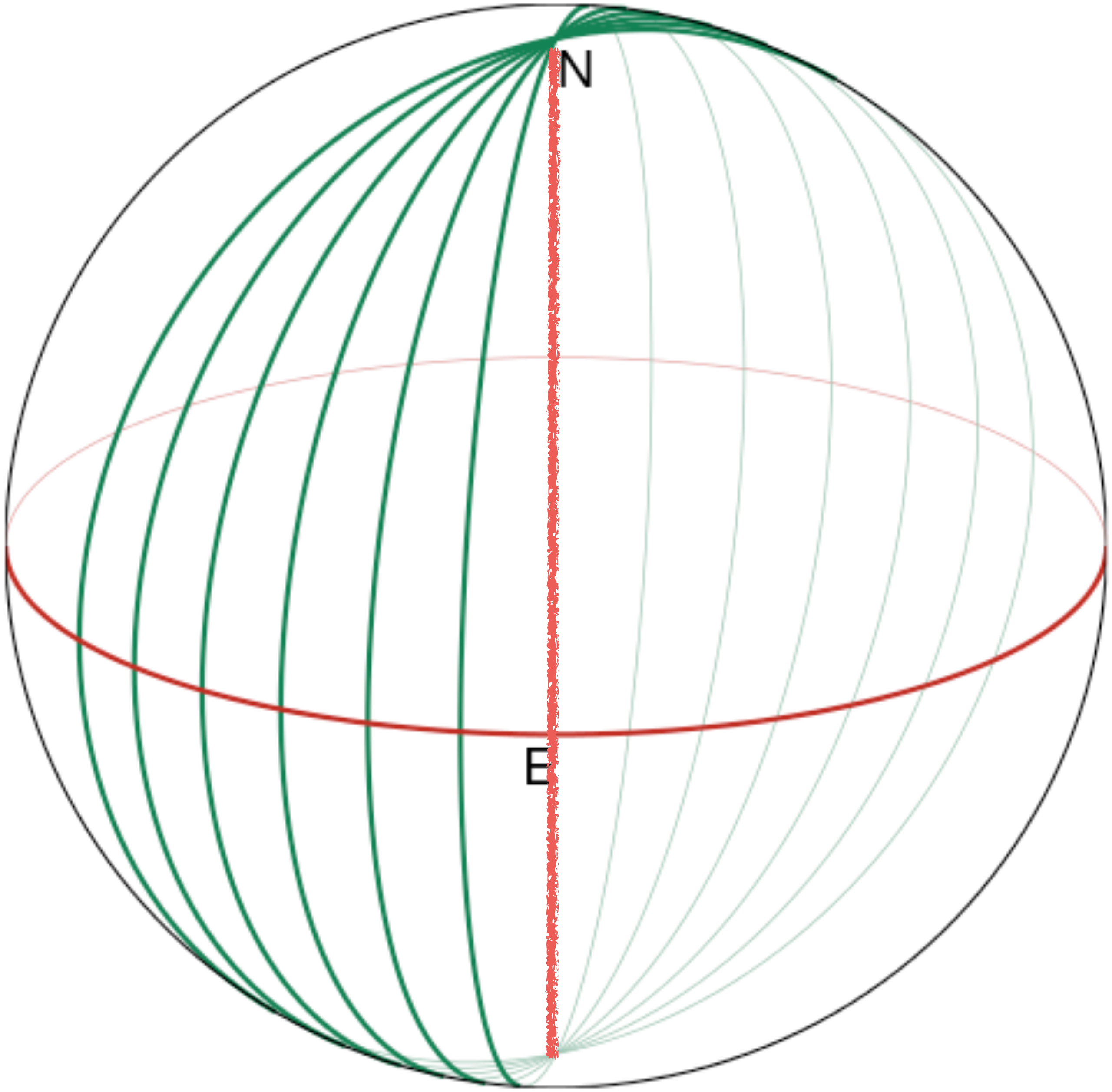
$$\Phi = 1/2, \;\; \mathcal{D} = \arccos(\Phi) = \pi/3, \;\; \gamma\tau = 2 \ln 2 \approx 1.39$$

Time for getting deexcited:

$$\mathcal{D} = \pi/2 \Rightarrow \tau = \infty!$$

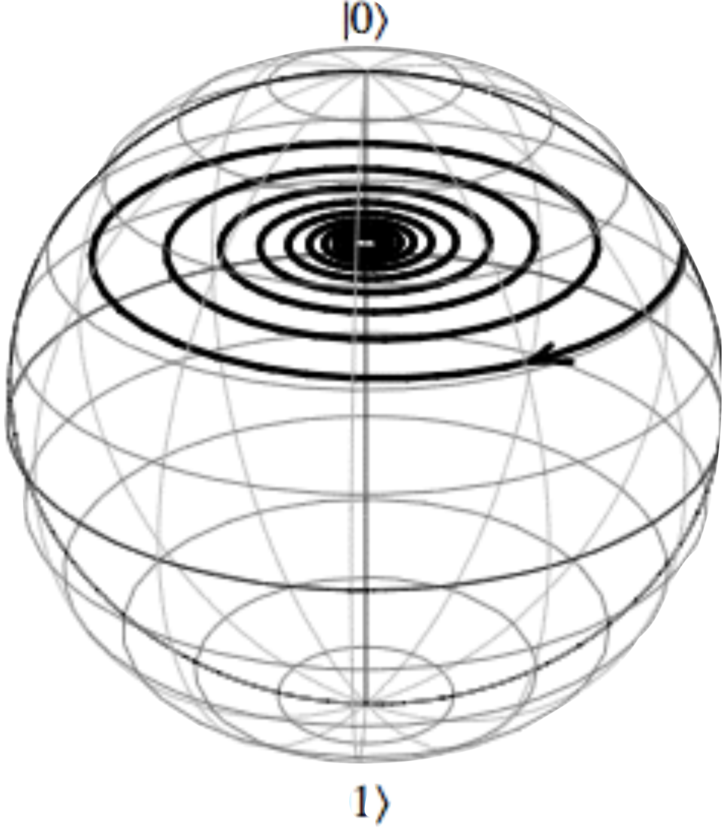
Quantum speed limit for physical processes: amplitude damping channel (3)

For pure states, the geodesics according to the Fubini-Study metric are segments of great circles of the sphere. The extension to mixed states, given by the Bures angle, adds other paths of the same length. The geometry of the Bures angle is therefore quite different from the usual Euclidean geometry on the Bloch sphere, since a diameter and a great half-circle have here the same length.



The picture shows the geodesics between an initial vector pointing up and a final vector pointing down.

Quantum speed limit for physical processes: Dephasing channel



The dephasing channel may be defined by the following set of equations:

$$|0\rangle|0\rangle_E \to e^{-i\omega_0 t}\left[\sqrt{P(t)}|0\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E\right],$$

$$|1\rangle|0\rangle_E \to e^{i\omega_0 t}\left[\sqrt{P(t)}|1\rangle|0\rangle_E - \sqrt{1-P(t)}|1\rangle|1\rangle_E\right],$$

$P(t) := (1+e^{-\gamma t})/2$ $\gamma(t) \to$ Dephasing rate

Note that the states $|0\rangle$ and $|1\rangle$ of the system do not change. However, a superposition like $(|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$ gets maximally entangled with two orthogonal states of the environment when $t\to\infty$, so phase information is lost on the system (even though the phase can still be recovered by joint measurements on S+E):

$$(1/\sqrt{2})\left(|0\rangle + e^{i\varphi}|1\rangle\right)|0\rangle_E \to (1/2)\left[|0\rangle\left(|0\rangle_E + |1\rangle_E\right) + e^{i\varphi}|1\rangle\left(|0\rangle_E - |1\rangle_E\right)\right]$$

The unitary evolution corresponding to the map is:

$\hat{U}_{S,E}(t) = e^{-i\omega_0 t\hat{\sigma}_z} e^{-i\theta(t)\hat{\sigma}_z\sigma_y^{(E)}}$ with $\theta(t) = \arccos\sqrt{P(t)}$.

$\hat{\sigma}_i \to$ Pauli matrices for system S

$\hat{\sigma}_i^{(E)} \to$ Pauli matrices for environment E

Quantum speed limit for physical processes: Dephasing channel

$$|0\rangle|0\rangle_E \rightarrow e^{-i\omega_0 t}\left[\sqrt{P(t)}|0\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E\right],$$

$$|1\rangle|0\rangle_E \rightarrow e^{i\omega_0 t}\left[\sqrt{P(t)}|1\rangle|0\rangle_E - \sqrt{1-P(t)}|1\rangle|1\rangle_E\right],$$

$P(t) := (1+e^{-\gamma t})/2$ $\gamma(t) \rightarrow$ Dephasing rate

Unitary evolution corresponding to the map:

$$\hat{U}_{S,E}(t) = e^{-i\omega_0 t\hat{\sigma}_z} e^{-i\theta(t)\hat{\sigma}_z\sigma_y^{(E)}} \quad \theta(t) = \arccos\sqrt{P(t)}$$

This is already a possible purification of the evolution. It is possible however to do better than this, by looking for a parametrization of the most general purification.

More general unitary evolution: $\hat{\mathcal{U}}_{S,E}(t) = \hat{u}_E(t)\hat{U}_{S,E}(t)$

Minimize $\mathcal{C}_Q(t)$ over all possible evolutions $\hat{u}_E(t)$. $\mathcal{C}_Q(t)$ depends only on

$$\hat{h}_E(t) := \frac{\hbar}{i}\frac{d\hat{u}_E^\dagger(t)}{dt}\hat{u}_E(t)$$

Set $\hat{h}_E(t) = \alpha(t)\hat{\sigma}_x^{(E)} + \beta(t)\hat{\sigma}_y^{(E)} + \gamma(t)\hat{\sigma}_z^{(E)}$

$\alpha(t)$, $\beta(t)$, $\gamma(t) \rightarrow$ Variational parameters

This is the most general transformation, in this case!

Quantum speed limit for physical processes: Dephasing channel

For simplicity, we consider here the special case $\omega_0 = 0$. One has then:

$$\mathcal{D} \leq \frac{1}{2}\sqrt{\langle \Delta \hat{Z}^2 \rangle}\, \arccos[\exp(-\gamma\tau/2)] \Rightarrow \gamma\tau \geq \ln\sec\left(2\mathcal{D}/\sqrt{\langle \Delta \hat{Z}^2 \rangle}\right)$$

Note that $\langle \Delta \hat{Z}^2 \rangle = 0 \Rightarrow$ Eigenstate of Z: no evolution

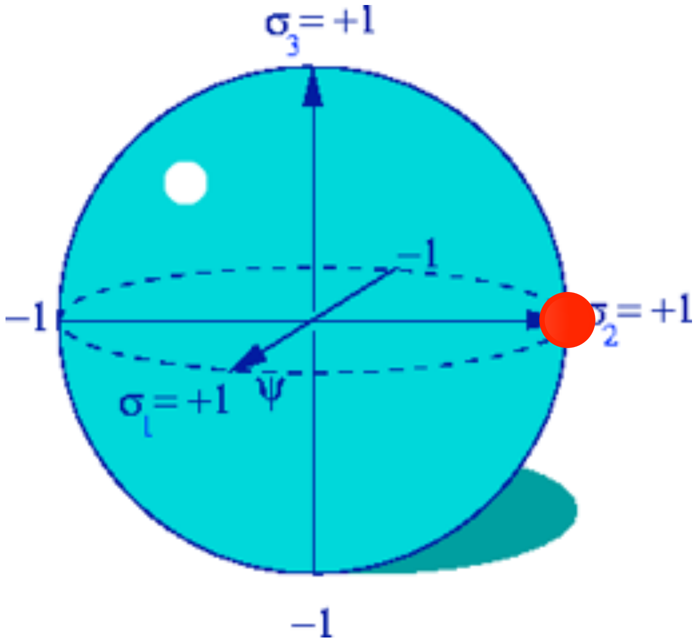
Maximum distance between states: $\sqrt{\langle \Delta \hat{Z}^2 \rangle}\pi/4$

Pure states with $\langle \Delta \hat{Z}^2 \rangle = 1 \Rightarrow$ Bound is saturated

Interpretation: These states are represented by vectors in the equatorial plane of the Bloch sphere.

Since $\omega_0 = 0$, evolution is along geodesic of Bloch sphere:

$$(|0\rangle + |1\rangle)/\sqrt{2} \rightarrow (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$$



Quantum speed limit for physical processes: Dephasing channel

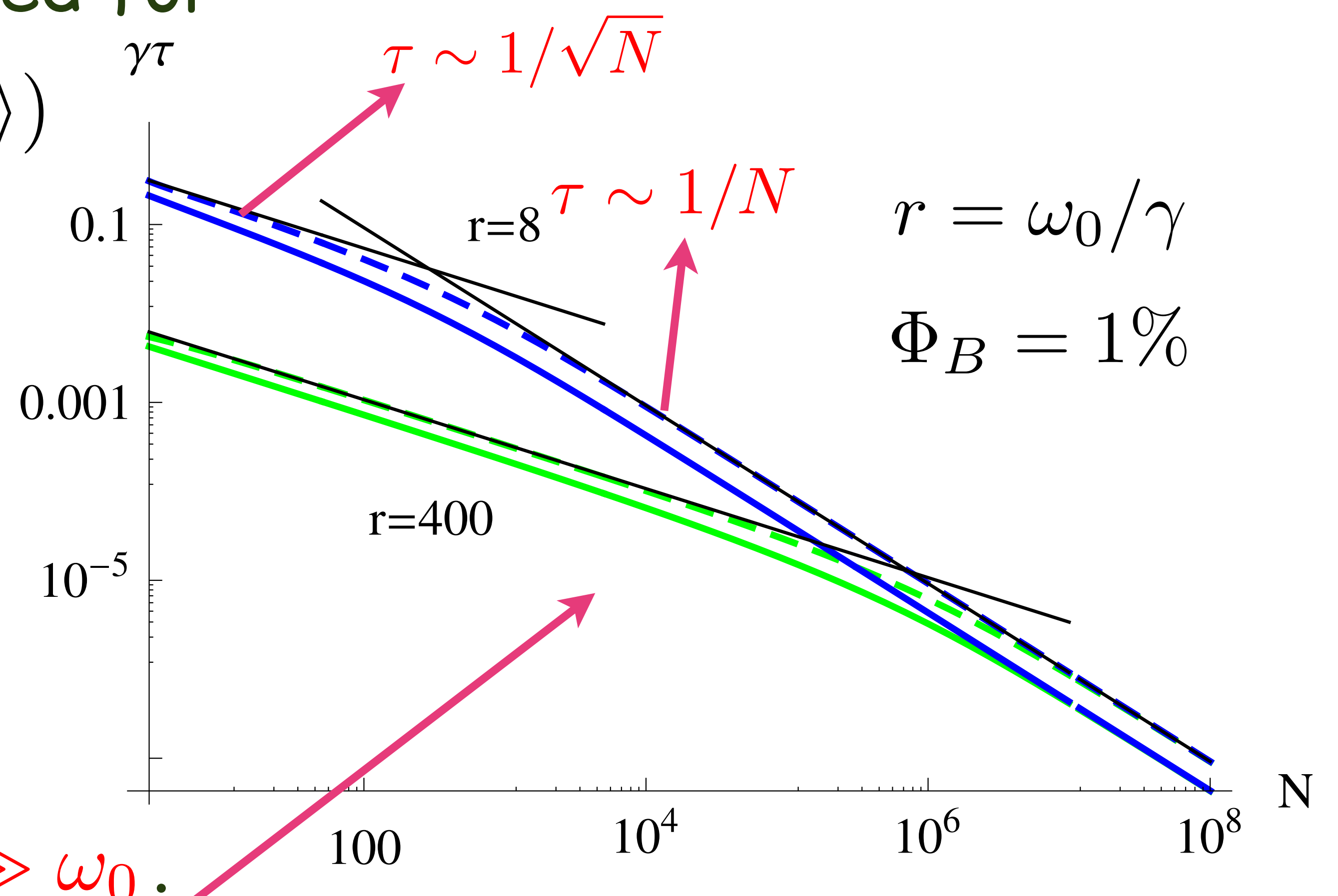
N-qubit system, each interacting with its own dephasing reservoir

Try $\hat{h}_E(t) = \sum_i [\alpha(t)\hat{X}_i^{(E)} + \beta(t)\hat{Y}_i^{(E)} + \gamma(t)\hat{Z}_i^{(E)}]$, where $\hat{X} \equiv \sigma_x,\ \hat{Y} \equiv \sigma_y,\ \hat{Z} \equiv \sigma_z.$

Lower bound scales as $\tau \sim 1/N$. Attained for

GHZ states $(1/2)(|0\dots0\rangle + e^{i\phi}|1\dots1\rangle)$

$$\boxed{\Phi_B[\hat{\rho}(0),\hat{\rho}(t)] = \frac{1+e^{-N\gamma\tau}\cos 2N\omega_0\tau}{2}}$$



Separable states:

Lower bound scales as $\tau \sim 1/\sqrt{N}$ for $\gamma\sqrt{N} \ll \omega_0$ and as $\tau \sim 1/N$ for $\gamma\sqrt{N} \gg \omega_0$.

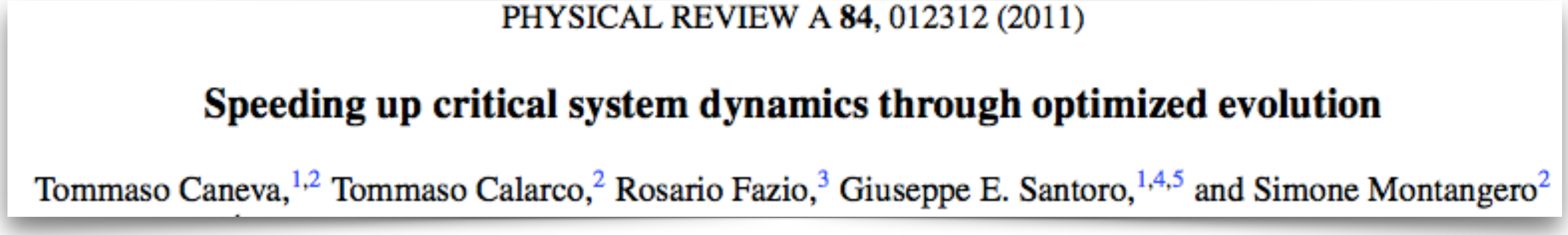
Product state, qubits initially in state

$$(|0\rangle + |1\rangle)/\sqrt{2} \Rightarrow \Phi_B = \frac{1}{2^N}\left(1+e^{-\gamma\tau}\cos 2\omega_0\tau\right)^N$$

Lower bound: full lines

Exact solution: dashed lines

Quantum speed limit and quantum control



PHYSICAL REVIEW A **84**, 012312 (2011)

Speeding up critical system dynamics through optimized evolution

Tommaso Caneva,[1,2] Tommaso Calarco,[2] Rosario Fazio,[3] Giuseppe E. Santoro,[1,4,5] and Simone Montangero[2]

Goal: Maximize fidelity $|\langle\psi(T)|\psi_G\rangle|^2$, for fixed T starting with ground state of Hamiltonian and having as target the ground state of modified Hamiltonian (as in adiabatic quantum computation).

Control function optimized numerically (Krotov algorithm)

Landau-Zener:

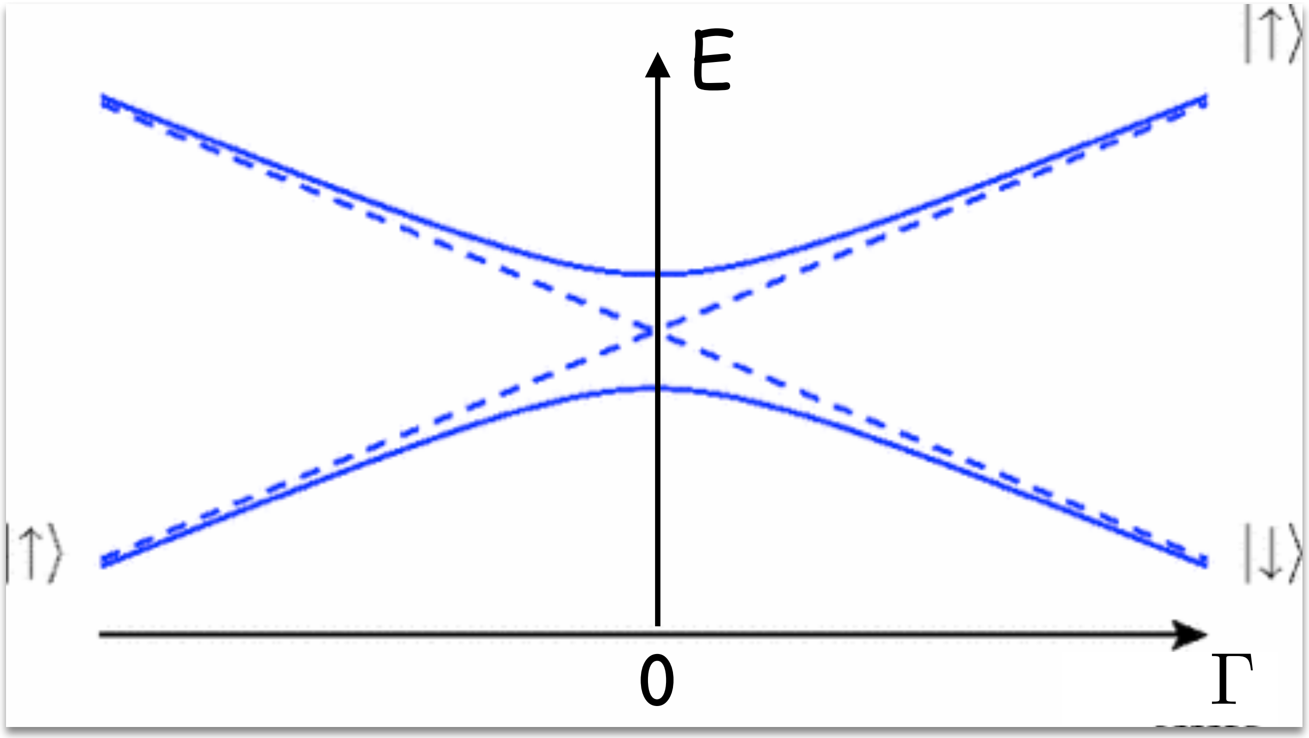
Control

$$\hat{H}(t)=\frac{\hbar}{2}\left[\Gamma(t)\hat{\sigma}_z+\omega_0\hat{\sigma}_x\right]$$

Initial state: GS with $\Gamma(-T/2)=-\Gamma_0$

Target state: GS with $\Gamma(T/2)=\Gamma_0$

Fast change in beginning and end: $\Gamma=0$ in between



Better than adiabatic change!

Quantum speed limit and unbounded $\Gamma(t)$

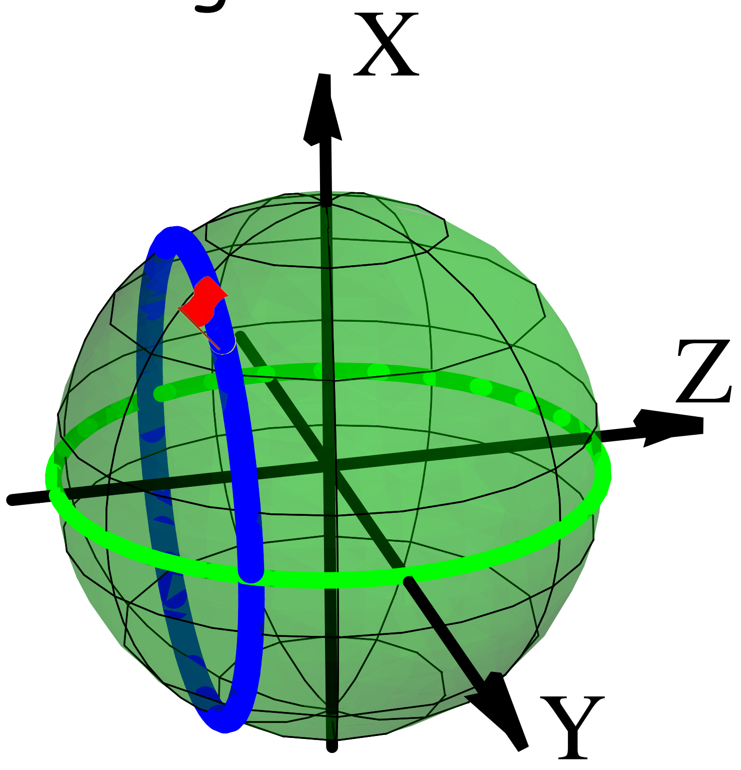
$$\hat{H}(t)=\frac{\hbar}{2}\left[\Gamma(t)\hat{\sigma}_z+\omega_0\hat{\sigma}_x\right] \qquad \frac{d\vec{r}}{dt}(t)=\vec{\Gamma}\times\vec{r}(t),\ \vec{\Gamma}(t)=\Gamma(t)\hat{z}+\omega_0\hat{x}$$

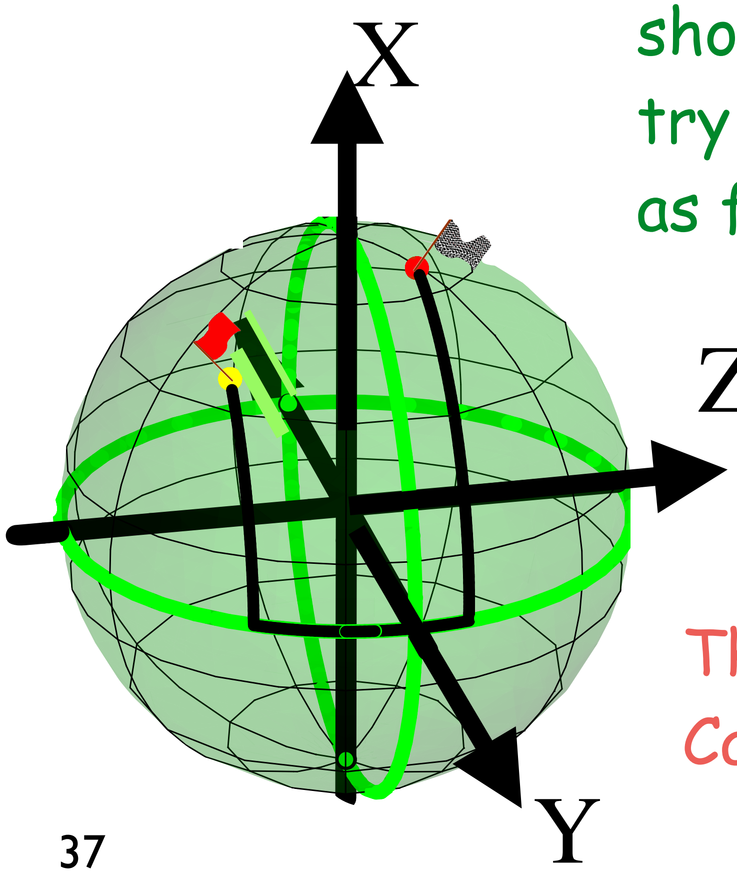
From $\arccos\sqrt{F\left(|\psi_0\rangle,|\psi_f\rangle\right)}\leq\int_0^\tau\frac{\Delta E(t)}{\hbar}dt\leq\frac{\Delta E_{\mathrm{MAX}}}{\hbar}\tau$

$\tau\geq\frac{\arccos|\langle\psi_0|\psi_f\rangle|}{\Delta E_{\mathrm{MAX}}}\to 0$ if $\Delta E\to\infty$

Going from initial to final state in the shortest possible time: try to reach a geodesic as fast as possible!

Bound is tight!





This is the result of Caneva et al.!

Conclusions

In this series of lectures, we introduced basic notions of quantum metrology, and showed that quantum mechanics helps to improve the precision in the estimation of parameters. New developments regarding parameter estimation in open systems have been discussed. We have illustrated these ideas by considering the precision limits in the estimation of phases in a noisy optical interferometer, or yet of a small force acting on a damped harmonic oscillator. We have also shown that the methods of quantum metrology allow a very general approach to the quantum speed limit, allowing the extension of the energy-time uncertainty relation to open systems. As a matter of fact, quantum metrology is a very active field nowadays. Experiments involving the detection of tiny magnetic or electric fields have already been implemented. A possible application of these ideas is related to the recent detection of gravitational waves. This involved comparing the relative lengths of the two arms of an interferometer to within 1/10,000 the diameter of a proton. An even better precision could be obtained through the use of squeezed states, already demonstrated in the gravitational antennas of the LIGO project, as discussed in the first lecture.

Collaborators

Gabriel Bié

Marcio Taddei

Camille Latune

Bruno Escher

Nicim Zagury

Ruynet Matos Filho