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Quantum limit for the measurement of a classical force coupled to a noisy quantum-mechanical oscillator

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Precise measurements of tiny forces and displacements play an important role in science and technology. The precision of recent experiments, while beginning to reach the limits imposed by quantum mechanics, is necessarily spoiled by the unavoidable influence of noise. Here we obtain a quantum limit for the uncertainty in the estimation of the amplitude of a classical force with arbitrary time dependence, acting on a noisy quantum-mechanical oscillator. We determine the best initial state of the oscillator and the best measurement procedure, thus getting a rigorous and useful benchmark for experiments aiming to detect extremely small forces.

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I. INTRODUCTION

Several areas of science and technology rely on the capacity to measure tiny forces and displacements [1–8]. The sensitivity of recent experiments has attained extremely high levels [9–12], so that the limits imposed by quantum mechanics start playing an important role [13–16]. Quantum metrology [17,18] yields, for noiseless systems, useful expressions for the ultimate quantum precision limits. However, for these extreme precisions, the unavoidable environment-induced noise must be considered. Nevertheless, the determination of the ultimate precision limit in the presence of noise is still a challenging problem in quantum mechanics. Here we determine the quantum limit for the uncertainty in the estimation of the amplitude of a classical force, whose waveform is assumed to be arbitrary but known [19], acting on a noisy quantum-mechanical oscillator, which is used as a probe for the measurement of the force through a sequence of discrete measurements. This harmonic oscillator could stand, for instance, for a trapped ion under an applied electric field, or a mesoscopic mechanical slab acted upon by a weak force, or yet a mode of an electromagnetic field fed by an electric current.

The problem of estimating the amplitude of a classical force is important in many areas of science, and most notably in the domains of nanophysics and gravitational waves. This problem was discussed in the pioneer works of Braginsky and collaborators [1,3] and Caves *et al.* [2]. Bounds on the uncertainty in the amplitude of a force are especially useful when the time dependence of the force is previously known, as may be the case in nanophysics, or when the model of the process leads to a specific waveform, as in source-dependent gravitational waves [20].

Our solution, which comprises a tight analytical bound for the uncertainty in the estimation of the force amplitude, leads to the best probe state, for a given average energy of the oscillator, and to a measurement procedure that attains, in the asymptotic high-energy limit, the so-called "potential sensitivity", which defines a level of sensitivity in the estimation of a force in the presence of thermal noise that cannot be surpassed by any measurement strategy [3]. Furthermore, our exact result leads to corrections to the potential sensitivity, for finite energies. As a specific application, we derive a precise lower bound for the uncertainty in the estimation of a resonant force acting on a trapped ion in a realistic noisy setup [10].

The results in this paper are obtained by applying the method proposed in Refs. [21–23]. There, one derives a lower bound to the precision of parameter estimation for a generic noisy system S by adding an environment R that leads to the correct reduced dynamics of the system alone. This amounts to a purification of the original dynamics, turning the nonunitary evolution of the system into a unitary one in an extended space, for which a lower bound to the precision of estimation can be computed analytically. Since measurements on both the system and the environment should not lead to less information about the parameter than measurements on the system alone, the lower bound to the precision, obtained through this purification procedure, should not be larger than the lower bound corresponding to S.

One gets therefore, by this method, a lower bound for the precision of estimation based on measurements on S, which depends on the choice of environment and is not necessarily tight. It was shown however in Ref. [21] that it is always possible to choose an extension of the original Hilbert space such that this bound is actually attainable through measurements only on S. In this case, measurements on system plus environment yield redundant information with respect to measurements on the system alone. A systematic procedure for obtaining this ideal extension was presented in Ref. [23]. One may get then, through this procedure, a tight precision limit for noisy systems. This method has been successfully applied to solve important problems in quantum metrology involving optical interferometry with photon losses [21] and phase diffusion [23], atomic spectroscopy [21], and the quantum speed limit [24]. It has also been used to address problems involving multiparameter estimation [25].

This article is organized as follows: Sec. II summarizes the main results of quantum metrology for unitary evolutions of pure states, introducing the concepts of Crámer-Rao bound and quantum Fisher information, which play an important role in quantum metrology, since the lower bound for the precision in the estimation of a parameter is inversely proportional to the square root of the quantum Fisher information corresponding to that parameter.

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Section III discusses the problem of quadrature measurements, during a given time interval, on a forced noisy harmonic oscillator, prepared in an initial Gaussian state, and derives the corresponding Fisher information, which leads to a precision bound for the estimation of the force amplitude.

In Sec. IV, we consider, in an extended space (harmonic oscillator plus environment), a unitary evolution that leads to the correct reduced dynamics of the forced noisy oscillator. Using a variational technique, we obtain an upper bound to the quantum Fisher information for the forced noisy harmonic oscillator. We show that this upper bound actually coincides, for minimal-uncertainty states, with the Fisher information obtained in the previous section, and therefore it leads to an attainable lower bound for the precision in the estimation of the amplitude of the force, probed by the noisy harmonic oscillator. This is the best lower bound, for a given duration of the measurement. We also determine the corresponding best initial state and the best detection procedure, for this time-dependent measurement procedure.

Section V shows that the ultimate limit for the uncertainty is obtained through sequential measurements made at optimal time intervals, leading to the potential sensitivity when the average energy of the oscillator goes to infinity, and yielding corrections to this expression for finite values of the average energy.

Section VI considers the role of measurement accuracy, while Sec. VII discusses some applications of these results and demonstrates a striking discontinuity of the optimal time interval as a function of the energy of the oscillator or the temperature of the bath.

The Appendices include detailed calculations of some expressions displayed in the main text of the article. Our conclusions are summarized in Sec. VIII.

II. CRÁMER-RAO BOUND AND FISHER INFORMATION

The precise detection of a classical force is an example of the general problem of parameter estimation. A typical procedure consists in sending a probe in a known initial state through some parameter-dependent physical process and measuring the final state of the probe, estimating then from this measurement the value of the parameter. The precision generally depends on the initial state of the probe, on the measurement and estimation procedure, on the dynamical process, and on the amount of resources used in the measurement (quantified for instance by the number of probes or the energy of each probe). It is given by the Crámer-Rao bound [26,27], which relates the uncertainty δx in the estimation of a parameter x to the Fisher information [28] $\mathcal{F}(x)$ defined in terms of the conditional probability density $\mathcal{P}(\xi|x)$ of getting the outcome ξ of the measurement when the value of the parameter is x:

$$\delta x = \sqrt{\langle (x_{\text{est}} - x_{\text{true}})^2 \rangle} \geqslant \frac{1}{\sqrt{\nu \mathcal{F}(x_{\text{true}})}}, \tag{1}$$

where

$$\mathcal{F}(x) = \int d\xi \left\{ \frac{1}{\mathcal{P}(\xi|x)} \left[\frac{\partial \mathcal{P}(\xi|x)}{\partial x} \right]^2 \right\}, \tag{2}$$

x_{est} is the estimated value of the parameter for a possible measurement result, x_{true} is the true value of the parameter, ν is the number of repetitions of the experiment and the average in Eq. (1) is taken over all possible measurement results. The above expression holds for unbiased estimators, for which $\langle x_{\text{est}} \rangle = x$. In general, the lower bound in Eq. (1) is tight for $\nu \to \infty$; however, if $\mathcal{P}(\xi|x)$ is Gaussian the bound is attainable for any $\nu \geqslant 1$ [26–28]. Better precision is obtained upon increasing the Fisher information or the number of repetitions.

Quantum mechanics imposes restrictions on the precision of the estimation, since two outgoing states corresponding to two different values of the parameter are not necessarily distinguishable, and measurements must conform to quantum constraints. In particular, for a measurement procedure corresponding to a positive operator-valued measure (POVM) $\{\hat{E}_\xi\}$, with $\int d\xi\, \hat{E}_\xi = \hat{1}$, one has $\mathcal{P}(\xi|x) = \text{Tr}[\hat{\rho}(x)\hat{E}_\xi]$, where $\hat{\rho}(x)$ is the parameter-dependent density matrix of the probe. On the other hand, quantum features, such as entanglement and squeezing, help to increase the estimation accuracy beyond the standard limits [29–36], yielding better precision for the same amount of resources.

The application of the Crámer-Rao bound to quantum theory was initiated by Helstrom [17] and Holevo [18] and further investigated by Braunstein and Caves [37]. The maximization of the Fisher information in Eq. (2) over all possible measurement procedures, leading to the so-called quantum Fisher information, yields, in the noiseless case, a simple expression for the corresponding uncertainty (1). This result has been applied to many different systems [30,35,38]. If the initial state of the probe is a pure state $|\psi_0\rangle$ and the state of the outgoing probe is $|\psi(x)\rangle = \hat{U}(x)|\psi_0\rangle$, where $\hat{U}(x)$ is an x-dependent unitary operator, then the corresponding quantum Fisher information is four times the variance, calculated in the state $|\psi_0\rangle$, of the operator

$$\hat{\mathcal{H}}(x) := i \frac{d\hat{U}^\dagger(x)}{dx} \hat{U}(x); \tag{3}$$

that is,

$$\mathcal{F}_Q(x) = 4\langle \psi_0 | [\Delta \hat{\mathcal{H}}(x)]^2 | \psi_0 \rangle. \tag{4}$$

However, the estimation of parameters in the presence of noise poses formidable challenges. Only for very special situations is it possible to derive analytic lower bounds for the uncertainty [39–44]. We show now that the important problem of estimation of the amplitude of a classical force acting on a noisy harmonic oscillator can be analytically solved.

III. QUADRATURE MEASUREMENTS ON THE NOISY HARMONIC OSCILLATOR

The Hamiltonian of a harmonic oscillator under the action of a classical force is given in terms of dimensionless variables by $\hat{H}_S/\hbar\omega = (\hat{P}^2 + \hat{X}^2)/2 - F\zeta(t)\hat{X}$, where, in terms of the momentum $\hat{p}$, the position $\hat{x}$, the mass m, the oscillation frequency ω, and the force amplitude f, the dimensionless variables are defined by $\hat{P} = \hat{p}/\sqrt{m\hbar\omega}$, $\hat{X} = \hat{x}\sqrt{m\omega/\hbar}$, so that $[\hat{X},\hat{P}] = i$, and $F = f/\sqrt{\hbar m\omega^3}$. Here we consider that the oscillator is submitted to the force $f\zeta(t)$ and we assume that $\zeta(t)$, the time variation of the force, is already known, and such that Max $|\zeta(t)| = 1$. The aim here is the estimation of f.

Setting $\hat{X} = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$ and $\hat{P} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, so that $[\hat{a},\hat{a}^\dagger] = 1$, the Hamiltonian in the interaction picture is given by

$$\hat{H}_I(t) = -\hbar\omega F\zeta(t)(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})/\sqrt{2}, \tag{5}$$

with $\hat{H}_0 = \hbar\omega/2(\hat{P}^2 + \hat{X}^2) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. The corresponding unitary evolution operator is, in the rotating frame (neglecting a global phase factor),

$$\hat{U}_I(t,0) = \exp\left\{iF\int_0^t dt'[\alpha(t')\hat{a} + \alpha^*(t')\hat{a}^\dagger]\right\}, \tag{6}$$

with

$$\alpha(t) = (\omega/\sqrt{2})\zeta(t)e^{-i\omega t}. \tag{7}$$

Physical insight into the dynamics of the forced harmonic oscillator in the presence of thermal noise can be obtained from the corresponding Heisenberg-Langevin equation for the creation operator in the interaction picture:

$$d\hat{a}/dt = i\omega F\zeta(t)e^{i\omega t}/\sqrt{2} - \gamma\hat{a}/2 + \hat{f}_\gamma(t), \tag{8}$$

where $\gamma/2$ is the friction coefficient and $\hat{f}_\gamma(t)$ is a fluctuation force, with $\langle\hat{f}_\gamma(t)\rangle = 0$, $\langle\hat{f}_\gamma(t)\hat{f}_\gamma^\dagger(t')\rangle = \gamma(n_T + 1)\delta(t-t')$, $\langle\hat{f}_\gamma^\dagger(t)\hat{f}_\gamma(t')\rangle = \gamma n_T\delta(t-t')$, $\langle\hat{f}_\gamma(t)\hat{f}_\gamma(t')\rangle = \langle\hat{f}_\gamma^\dagger(t)\hat{f}_\gamma^\dagger(t')\rangle = 0$, where $n_T = [\exp(\hbar\omega/k_BT) - 1]^{-1}$ is the average number of thermal excitations at temperature T and k_B is the Boltzmann constant. Integration of Eq. (8) yields

$$\hat{a}(t) = \hat{a}(0)e^{-\gamma t/2} + iFD(t)/\sqrt{2} + \int_0^t dt'\hat{f}_\gamma(t')e^{\gamma(t'-t)/2}, \tag{9}$$

with

$$D(t) \equiv |D(t)|e^{i\phi_t} = \omega\int_0^t dt'\zeta(t')e^{i\omega t'}e^{-\gamma(t-t')/2}. \tag{10}$$

From Eq. (9) one gets that, in the presence of friction, the effect of the force is to displace the state in phase space by $F|D(t)|$ along the quadrature corresponding to the generalized momentum operator

$$\hat{P}(\phi_t) \equiv [\hat{a}\exp(-i\phi_t) - \hat{a}^\dagger\exp(i\phi_t)]/i\sqrt{2}. \tag{11}$$

Equation (9) also shows that the evolution of the variance of the operator $\hat{X}(\theta) = [\hat{a}\exp(-i\theta) + \hat{a}^\dagger\exp(i\theta)]/\sqrt{2}$ is

$$\langle[\Delta\hat{X}(\theta)]^2\rangle_t = \eta\langle[\Delta\hat{X}(\theta)]^2\rangle_0 + (2n_T + 1)(1-\eta)/2, \tag{12}$$

where the index t stands for the value of the variance at time t, and $\eta \equiv \exp(-\gamma t)$.

If the initial state of the harmonic oscillator is Gaussian, so is the final state $\hat{\rho}_t \equiv \hat{\rho}(t)$, since the oscillator is interacting with a thermal reservoir and the displacement is a Gaussian operation. Then the Fisher information for the estimation of F, corresponding to a generalized-momentum measurement on the final state of the oscillator, is

$$\mathcal{F}_P(F) = \int dP\frac{1}{\langle P|\hat{\rho}_t|P\rangle}\left(\frac{\partial\langle P|\hat{\rho}_t|P\rangle}{\partial F}\right)^2 = \frac{|D(t)|^2}{\langle[\Delta\hat{P}(\phi_t)]^2\rangle_t}, \tag{13}$$

where $|P\rangle$ is an eigenstate of $\hat{P}(\phi_t)$. Equations (12) and (13) imply that, for an initial minimum-uncertainty state in $X(\phi_t)$ and $P(\phi_t)$,

$$\mathcal{F}_P(F) = \frac{4|D(t)|^2\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0}{\eta + 2(2n_T+1)(1-\eta)\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0}, \tag{14}$$

where $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0 = 1/\{4\langle[\Delta\hat{P}(\phi_t)]^2\rangle_0\}$ is the variance of $\hat{X}(\phi_t)$ in the initial state.

We show now that this expression coincides with the quantum Fisher information for the class of initial states considered above, and that no other class of states yields a larger quantum Fisher information for the estimation of the force.

IV. QUANTUM FISHER INFORMATION FOR THE NOISY OSCILLATOR

We derive in the following section, by applying the purification method proposed in Refs. [21–23], the quantum Fisher information for a harmonic oscillator in contact with a zero-temperature environment ($T = 0$), which admits a simple physical picture that clarifies the main ingredients of our method. The more general condition $T \neq 0$ is discussed in the subsequent section.

A. Zero-temperature environment

A simple physical picture of the purification approach is obtained by considering the harmonic oscillator S as a mode of the electromagnetic field. The dissipation due to the interaction with the environment R is modeled by the beam splitter B_1, shown in Fig. 1. This device deflects photons into mode **b** (which plays the role of R). If the transmissivity of this beam splitter is $\eta = \exp(-\gamma t)$, then the evolution of the variances of the quadratures of the electromagnetic field is precisely that given by Eq. (12), when $n_T = 0$, which

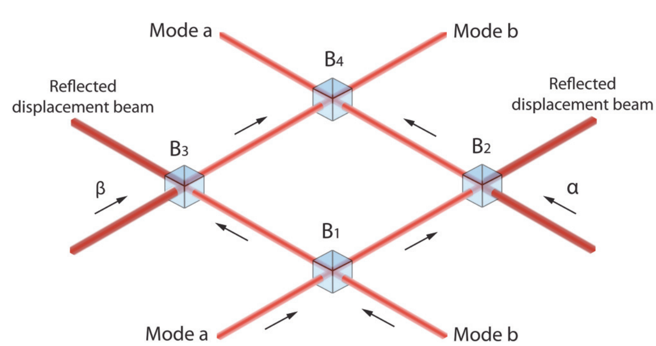

FIG. 1. (Color online) Beam-splitter model for the coupling with the environment. The incoming beam, in mode **a**, corresponds to the harmonic oscillator. Beam splitter B_1 induces photon losses in the incoming beam, as it deflects photons into mode **b**, which corresponds to the environment. Beam splitter B_2, with transmissivity $\mathcal{T} \to 0$, is used to displace the field in mode **a**, through the injection of a high-intensity coherent state with amplitude α, such that the product $\alpha\sqrt{\mathcal{T}}$ is finite. Beam splitter B_3 is used, analogously to B_2, to displace the field in the environment, upon injection of a coherent state with amplitude β. Beam splitter B_4 decouples modes **a** and **b**, allowing individual measurements on each mode.

motivates this beam-splitter picture of the dissipation process. The displacement is induced by a coherent state sent into a second beam splitter B_2, with transmissivity $\mathcal{T}$ going to zero at the same time that the amplitude α of the coherent state goes to infinity, the product $\sqrt{\mathcal{T}}\alpha$ remaining finite. In Appendix A, it is shown that these two operations yield an evolution for S alone equivalent to the one derived from the master equation corresponding to this problem. The initial state of $S+R$ is $|\psi_0\rangle_S|0\rangle_R$, the environment (mode **b**) being initially in the vacuum state.

After these two operations, the two-mode state becomes $|\psi(t)\rangle = e^{iF|D(t)|\hat{X}_S(\phi_t)}\hat{B}_1|\psi_0\rangle_S|0\rangle_R$, where $\hat{B}_1 = e^{\theta_1(t)(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})}$ is the beam-splitter operator acting on modes **a** and **b**, with $\cos\theta_1(t) = \sqrt{\eta}$; $\hat{a}$ and $\hat{b}$ are the annihilation operators corresponding to modes **a** and **b**, respectively; $\hat{X}_S(\phi_t)$ is a quadrature operator for mode **a**. The $\hat{X}_S$-dependent exponential displaces the generalized momentum $\hat{P}(\phi_t)$ in conformity with Eq. (9).

In general, measurements on this extended system yield more information on the parameter than measurements on S alone. In order to reduce the nonredundant information about F in $S+R$, we use the result that any two purifications of a density operator can be related by a unitary transformation acting on the environment alone [45]. More specifically, we displace the field in mode **b** along the same quadrature as the one displaced in mode **a**. This is implemented, as shown in Fig. 1, by sending a coherent state with amplitude β on beam splitter B_3, which has a vanishingly small transmittance, as was the case for B_2. This is a local operation on R, which does not affect S. The evolution operator in $S+R$ takes then the form

$$\hat{U}_{SR}(t) = e^{iFG(t)\hat{X}_R(\phi_t)}e^{-iF|D(t)|\hat{X}_S(\phi_t)}\hat{B}_1, \tag{15}$$

where $\hat{X}_R(\phi_t)$ is the quadrature rotated by an angle ϕ_t from the position quadrature corresponding to mode **b** in the interaction picture.

Inserting Eq. (15) into Eqs. (3) and (4), we get the respective quantum Fisher information:

$$\mathcal{F}_Q^{SR}[G(t)] = [-G(t)\sqrt{1-\eta} + |D(t)|\sqrt{\eta}]^2 4\langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0 + 2[|D(t)|\sqrt{1-\eta} + G(t)\sqrt{\eta}]^2, \tag{16}$$

where the averages are calculated in the initial state of $S+R$, and we have used that $\langle[\Delta\hat{X}_R(\phi_t)]^2\rangle_0 = 1/2$. According to the above discussion, this should be an upper bound for the quantum Fisher information associated with S alone for any value of $G(t)$. The best upper bound is obtained by determining the function $G(t)$ that minimizes Eq. (16). The minimum is reached for

$$G_{\text{opt}}(t) = \frac{\sqrt{\eta(1-\eta)}\{\langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0 - 1/2\}}{(1-\eta)\langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0 + \eta/2}|D(t)|. \tag{17}$$

The corresponding minimum quantum Fisher information $\mathcal{F}_Q^{SR}[G_{\text{opt}}(t)]$ for $S+R$ coincides then precisely with Eq. (14) for a zero-temperature reservoir, with $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0 \equiv \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0$.

The generalization of the above procedure for $T \neq 0$ is discussed in the following section. It involves the addition of both a squeezing transformation on mode **a** and another environment mode **c**.

B. Thermal environment with arbitrary temperature

We show in Appendix B that a purification of the density operator of the harmonic oscillator in contact with a thermal bath at temperature T can be built with an environment consisting of two harmonic oscillators R_1 and R_2, corresponding respectively to modes **b** and **c** in the equivalent electromagnetic description. This purification involves three unitary evolutions, applied to the initial state of $S+R_1+R_2$. The first unitary corresponds to a beam-splitter-like interaction between the system S and the environment R_1, the second one corresponds to a two-mode squeezing-like interaction between the system S and the environment R_2, and the third one corresponds to a phase-space displacement in S space:

$$|\Psi(t)\rangle = e^{iF|D(t)|\hat{X}(\phi_t)}\hat{S}\hat{B}_1|\psi_0\rangle_S|0\rangle_{R_1}|0\rangle_{R_2}, \tag{18}$$

where both environments are taken initially in the ground state, and

$$\hat{B}_1 = e^{\theta_1(t)(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})}, \tag{19}$$

$$\hat{S} = e^{\theta_2(t)(\hat{a}^\dagger\hat{c}^\dagger - \hat{a}\hat{c})}, \tag{20}$$

with $\theta_1(t)$ and $\theta_2(t)$ given by

$$\theta_1(t) = \arccos\left[\sqrt{\frac{\eta}{n_T(1-\eta)+1}}\right], \tag{21}$$

$$\theta_2(t) = \text{arccosh}[\sqrt{n_T(1-\eta)+1}]. \tag{22}$$

Here $\hat{b}$ ($\hat{c}$) is the annihilation operator for the environment R_1 (R_2).

As in the previous section, we may use the freedom of choosing different purifications leading to the same reduced description in order to minimize the quantum Fisher information corresponding to $S+R_1+R_2$. In this extended space, the most general purification of the corresponding density operator is given by

$$|\Phi\rangle = e^{iF|D(t)|\hat{H}_{1,2}}|\Psi(t)\rangle, \tag{23}$$

where $\hat{H}_{1,2}$ is a Hermitian operator acting only on the environments R_1 and R_2, and $|\Psi(t)\rangle$ is defined in Eq. (18). In order to get the lowest upper bound for the quantum Fisher information of the system, the operator $\hat{H}_{1,2}$ should be chosen properly in order to minimize the quantum Fisher information corresponding to $S+R_1+R_2$.

A possible choice of the operator $\hat{H}_{1,2}$, aimed to erase at least part of the nonredundant information on the value of the force F in $|\Phi\rangle$, as compared with the same information in $\hat{\rho}_t$, is $\hat{H}_{1,2} = \lambda_1\hat{X}_{R_1}(\phi_t) + \lambda_2\hat{X}_{R_2}(\phi_t)$, where $\hat{X}_{R_1}(\phi_t)$ [$\hat{X}_{R_2}(\phi_t)$] is the rotated quadrature operator of the oscillator in R_1 (R_2) space. This choice is based on physical insight on the enlarged unitary process, as discussed in the following. Since $|\Phi\rangle$ is an entangled state in $S+R_1+R_2$, the effect of a phase-space displacement of S may be transferred to $S+R_1+R_2$ through an F-independent disentanglement operation, which would be an ingredient of a measurement on $S + R_1 + R_2$. As shown in Appendix B, this operation is implemented by the unitary operator $\hat{B}_1^\dagger \hat{S}^\dagger$, which does not change the quantum Fisher information of $S + R_1 + R_2$. The resulting effect is to displace all oscillators separately along the quadrature $\hat{X}_i(\phi_t)$, $i = S, R_1, R_2$, in each space. This implies that, with $\hat{H}_{1,2} = \lambda_1 \hat{X}_{R_1}(\phi_t) + \lambda_2 \hat{X}_{R_2}(\phi_t)$ and convenient values of λ_1 and λ_2, it is possible to erase at least part of the nonredundant information in $|\Psi(t)\rangle$.

Minimization of the quantum Fisher information of $S + R_1 + R_2$ with respect to the parameters λ_1, λ_2 leads, as shown in Appendix C, to the Fisher information (14), obtained in the previous section for an initial minimum-uncertainty state of the probe, and for a generalized-momentum measurement.

This shows that the upper bound obtained by this minimization procedure is actually attained for initial minimum-uncertainty states in $X(\phi_t)$ and $P(\phi_t)$, implying that it is the quantum Fisher information of S alone, for these initial states. It also implies that no other class of states yields a larger quantum Fisher information for the estimation of the force. Furthermore, for these states, the best measurement for the estimation of the force is a measurement of the quadrature $P(\phi_t)$.

Using Eq. (1) and the upper bound obtained with this approach, one gets, for any initial state of the harmonic oscillator, the following inequality for the uncertainty δf in the estimation of the force:

$$\delta f \geqslant \frac{\sqrt{m\hbar\omega^3}}{|D(t)|\sqrt{2\nu}} \sqrt{(1-\eta)(2n_T+1) + \frac{\eta/2}{\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0}}; \quad (24)$$

this limit being attainable, for any integer value of ν, by Gaussian minimum-uncertainty states in $\hat{X}(\phi_t)$ and $\hat{P}(\phi_t)$. The standard limit corresponds to $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0 = 1/2$, so that the right-hand side of Eq. (24) becomes $\sqrt{m\hbar\omega^3}[D(\eta)\sqrt{2\nu}]^{-1}[2n_T(1-\eta)+1]^{1/2}$.

The lossless case is obtained for $\gamma \to 0$; then, only the second term inside the square root is left. On the other hand, when $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0 \gg \eta/[(1-\eta)(2n_T+1)]$ the lower bound becomes $\sqrt{m\hbar\omega^3}[D(\eta)\sqrt{2\nu}]^{-1}\sqrt{(1-\eta)(1+2n_T)}$. Therefore, for sufficiently large $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0$, one gets an expression similar to the one for a coherent state (standard limit), no matter how small are the losses, but lowered by a factor $\sqrt{(1-\eta)(2n_T+1)/[2n_T(1-\eta)+1]}$, which becomes $\sqrt{1-\eta}$ when $n_T \ll 1$.

Since (24) is a monotonic function of the variance $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0$, one must maximize this quantity in order to minimize the lower bound on δf. For fixed average initial energy $\langle\hat{X}^2(\phi_t)\rangle_0 + \langle\hat{P}^2(\phi_t)\rangle_0 = 2E$, where E is the energy in units of $\hbar\omega$, and taking into account the physical restriction imposed by the Heisenberg uncertainty relation $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0\langle[\Delta\hat{P}(\phi_t)]^2\rangle_0 = a \geqslant 1/4$, one gets a system of two equations with two unknown parameters:

$$\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0\langle[\Delta\hat{P}(\phi_t)]^2\rangle_0 = a, \quad (25)$$

$$\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0 + \langle[\Delta\hat{P}(\phi_t)]^2\rangle_0 = 2E^*, \quad (26)$$

with $2E^* = 2E - \langle\hat{X}(\phi_t)\rangle_0^2 - \langle\hat{P}(\phi_t)\rangle_0^2$. The solutions are $\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0^{\pm} = E^* \pm \sqrt{E^{*2} - a}$. It follows that the maximum value of the variance in $\hat{X}(\phi_t)$ under the two constraints imposed above is

$$\langle[\Delta\hat{X}(\phi_t)]^2\rangle_0 \equiv E + \sqrt{E^2 - 1/4}, \quad (27)$$

reached when $a = 1/4$ and $\langle\hat{X}(\phi_t)\rangle_0 = \langle\hat{P}(\phi_t)\rangle_0 = 0$; that is, when the state is a minimum uncertainty state in $X(\phi_t)$ and $P(\phi_t)$, centered around the origin in phase space.

This implies that the state that maximizes the variance of $\hat{X}(\phi_t)$, under the constraints imposed above, is a squeezed ground state $|\psi_0\rangle = \hat{S}(\xi)|0\rangle$, with $\xi = -re^{2i\phi_t}$, $\hat{S}(\xi) = \exp[-\frac{r}{2}(e^{-2i\phi_t}\hat{a}^2 - e^{2i\phi_t}\hat{a}^{\dagger 2})]$, and $r = 1/2\ln[2(E + \sqrt{E^2 - 1/4})]$.

The above discussion implies that the equality in Eq. (24) is attained through generalized-momentum measurements on the oscillator, initialized in a generalized-momentum-squeezed ground state, and using the maximum-likelihood estimator [26–28]. Since the state of the oscillator remains Gaussian throughout the evolution, equality in Eq. (24) holds even for $\nu = 1$.

V. ULTIMATE LIMIT FOR FORCE ESTIMATION

If the noisy harmonic oscillator senses the force during a time t and then is submitted to a single measurement, it follows from Eq. (24) that, in the limit $\gamma t \gg 1$, the uncertainty in the estimation of the force becomes independent of t and of the initial state of the oscillator. This is due to the fact that, in this case, the measurement will be done effectively on the steady state of the probe, which is independent of its initial state. These considerations suggest that, if the unknown force acts during a time $t_{\rm tot}$, it is better to probe the force through a sequential-measurement procedure. After each time interval τ, one measures the generalized momentum of the probe system, and then one reinitializes the system in a new squeezed optimal state, repeating this procedure ν times, with $\nu = t_{\rm tot}/\tau$. This may be done with the scheme proposed in Ref. [46]: radiation pulses much shorter then the characteristic evolution time of the oscillator allow both the measurement of the proper quadrature and the preparation of the best squeezed state for each interval of time. The optimal time $\tau_{\rm opt} = t_{\rm tot}/\nu_{\rm opt}$ is determined so as to minimize the uncertainty in the estimate of the amplitude of the force. This strategy, even though experimentally challenging, eliminates the back action of the momentum measurements and leads to a lower bound that provides a benchmark for the uncertainty in the estimation of the force. Another approach to evading the measurement back action was recently proposed in Ref. [47].

The coefficient D in Eq. (24) depends on the instant of time when the oscillator starts sensing the force and on the elapsed time before a measurement is performed. This dependence can be made explicitly by redefining the time dependence of this coefficient so that

$$|D(t_1,t_2)|e^{i\phi_{t_1,t_2}} \equiv \omega e^{-\gamma t_2/2}\int_{t_1}^{t_2} dt\,\zeta(t)e^{t(\gamma/2+i\omega)}. \quad (28)$$

Here we assume for definiteness that the classical force is zero for $t \leqslant t_0$ and $t \geqslant t_f$, with $t_{\rm tot} = t_f - t_0$. Dividing the interval $[t_0,t_f]$ into ν equal parts, we can perform ν measurements. The first measurement is accomplished at $t_0+\tau$, the second one at $t_0+2\tau$, etc., where $\tau\equiv(t_f-t_0)/\nu$. Using Eq. (14), an upper limit to the quantum Fisher information that corresponds to an experiment where the state of the oscillator is initialized at the time t and an optimal measurement is performed at $t+\tau$ is given by

$$\mathcal{F}_Q^S(t;\tau)\leqslant\frac{2|D(t;t+\tau)|^2}{(1-\eta)(2n_T+1)+\frac{\eta/2}{\langle[\Delta\hat{X}_S(\phi_{t;t+\tau})]^2\rangle_t}},\qquad(29)$$

where $\eta=e^{-\gamma\tau}$, and $\langle[\Delta\hat{X}_S(\phi_{t;t+\tau})]^2\rangle_t$ denotes the variance of the quadrature $\hat{X}_S(\phi_{t;t+\tau})$ taken at the instant t. The right-hand side of Eq. (29) is attained for initial minimum-uncertainty Gaussian states of the harmonic oscillator, as discussed in Sec. IV.

We assume in the following that the operation that initializes the state of the quantum oscillator can be performed instantly or, at least, in a interval of time that is negligible when compared with any other characteristic time of the process.

We consider then a strategy of estimation that (i) initializes the state of the oscillator at t_0; (ii) lets the oscillator sense the force for a time interval τ; (iii) measures the system at $t_0+\tau$; (iv) initializes the state of the oscillator to a convenient one at $t_0+\tau$; (v) lets the oscillator sense the force for another interval τ; (vi) measures the system at $t_0+2\tau$; (vii) repeats the process until the last measurement at $t_f=t_0+\nu\tau$. For this sequential strategy, the corresponding quantum Fisher information is the sum of the quantum Fisher information given by Eq. (29) for each measurement step:

$$\mathcal{F}_Q(\tau)=\sum_{n=0}^{\nu-1}\mathcal{F}_Q^S(t_0+n\tau;\tau).\qquad(30)$$

The quantum Fisher information for the sequential strategy is a function of τ and of the variances of $\hat{X}_S(\phi_{t_0+n\tau;t_0+(n+1)\tau})$ for each n. The maximal variance is $\langle[\Delta\hat{X}_S(\phi_{t_0+n\tau;t_0+(n+1)\tau})]^2\rangle_{t_0+n\tau}=E+\sqrt{E^2-1/4}$, for a given average energy of the oscillator E (in units of $\hbar\omega$). It is required then to initialize, before each measurement, the harmonic oscillator in a convenient state. This implies that Eq. (30) is limited by

$$\mathcal{F}_Q(\tau,\mathcal{E})=\frac{2/(2n_T+1)}{1-(1-1/\mathcal{E})e^{-\gamma\tau}}\times\sum_{n=0}^{\nu-1}|D[t_0+n\tau;t_0+(n+1)\tau]|^2,\qquad(31)$$

where

$$\mathcal{E}=2(2n_T+1)(E+\sqrt{E^2-1/4}).\qquad(32)$$

To maximize $\mathcal{F}_Q(\tau,\mathcal{E})$, for given $\mathcal{E}$ and $\zeta(t)$, one must optimize τ. This is accomplished in general by numerical methods. However, in the regime when τ is much smaller than all characteristic times of the process, which includes $1/\gamma$, $1/\omega$, and the characteristic time of evolution of $\zeta(t)$, it is possible to determine an analytical expression for $\tau_{\rm opt}$ and for $\mathcal{F}_Q(\tau_{\rm opt},\mathcal{E})$. As shown in Appendix D, the optimal time $\tau_{\rm opt}$ is then well approximated by

$$\gamma\tau_{\rm opt}\simeq\left[\frac{24/\mathcal{E}}{1+4(\omega^2+\overline{\omega^2})/\gamma^2}\right]^{1/3},\qquad(33)$$

where $\overline{\omega^2}$ is the mean-square frequency of the applied force, defined by

$$\overline{\omega^2}=\frac{\int_{-\infty}^{\infty}|\tilde{\zeta}(\omega')|^2\omega'^2d\omega'}{\int_{-\infty}^{\infty}|\tilde{\zeta}(\omega')|^2d\omega'},\qquad(34)$$

and $\tilde{\zeta}(\omega')$ is the Fourier transform of $\zeta(t)$. A characteristic time for the evolution of the force may be defined as $(\overline{\omega^2})^{-1/2}$.

Let $\delta f_{\mathcal{E}}=\sqrt{m\hbar\omega^3/\mathcal{F}_Q(\tau_{\rm opt},\mathcal{E})}$ be the lower bound for the uncertainty in the force estimation corresponding to the sequential measurement process. Then, under the same conditions,

$$\delta f_{\mathcal{E}}\simeq\delta f_{\rm min}\left\{1+\frac{1}{8}\left[1+\frac{4(\omega^2+\overline{\omega^2})}{\gamma^2}\right]^{1/3}\left(\frac{3}{\mathcal{E}}\right)^{2/3}\right\}.\qquad(35)$$

The quantity

$$\delta f_{\rm min}:=\sqrt{m\hbar\omega\gamma(n_T+1/2)/\xi},\qquad(36)$$

where $\xi=\int_{t_0}^{t_f}dt\zeta^2(t)$, is the minimum uncertainty when $\mathcal{E}\to\infty$ and coincides with the probe "potential sensitivity" derived in Ref. [3], which measures the strength of the thermal fluctuation force acting on the oscillator. This defines a lower bound to the uncertainty in the estimate of f, valid for any measurement strategy. We have thus proven here that this bound is actually attainable asymptotically by a minimum-uncertainty squeezed state.

The regime of validity of Eq. (35) is obtained by analyzing Eq. (33), in which the conditions $\gamma\tau_{\rm opt}\ll1$ and $\omega\tau_{\rm opt}\ll1$ are fulfilled for $\mathcal{E}\gg1$ and for $\mathcal{E}\gg\omega/\gamma$, respectively, while $(\overline{\omega^2})^{1/2}\tau_{\rm opt}\ll1$, for $\mathcal{E}\gg(\overline{\omega^2})^{1/2}/\gamma$. Therefore, it is expected that Eq. (35) is a good approximation of $\delta f_{\mathcal{E}}$ for $\mathcal{E}\gg\max\{1,\omega/\gamma,(\overline{\omega^2})^{1/2}/\gamma\}$.

One should note that Eq. (35) is a general result, valid for any continuous square-integrable time-dependent force in the interval $[t_0,t_f]$. The dependence on the pulse shape is manifested through the constants $\overline{\omega^2}$ and ξ.

VI. EFFECT OF UNSHARP QUADRATURE MEASUREMENTS

Equation (13) in Sec. III describes an ideal quadrature measurement, of infinite precision: the state of the oscillator immediately after the measurement is an eigenstate of the corresponding quadrature operator. We assume now a more realistic measurement, modeled by the POVM [48]

$$\hat{E}(\bar{P})\equiv\left(2\pi\sigma_P^2\right)^{-1/2}e^{-[\bar{P}-\hat{P}(\phi)]^2/(2\sigma_P^2)},\qquad(37)$$

where $\hat{P}(\phi)$ is the generalized momentum operator defined by Eq. (11). The probability density of getting the measurement result $\bar{P}$, given that the dimensionless force amplitude is F, is

$$\mathcal{P}(\bar{P}|F) = \mathrm{Tr}[\hat{\rho}_t \hat{E}(\bar{P})] = \int_{-\infty}^{+\infty} dP \, \frac{e^{-(\bar{P}-P)^2/(2\sigma_P^2)}}{\sigma_P\sqrt{2\pi}} \langle P|\hat{\rho}_t|P\rangle, \quad (38)$$

where $|P\rangle$ are eigenstates of the observable $\hat{P}(\phi)$. The distribution $\mathcal{P}(\bar{P}|F)$ is a convolution of $\langle P|\hat{\rho}_t|P\rangle$ and a Gaussian distribution that depends on the sharpness of the measurement device. For an oscillator in a Gaussian state one has

$$\langle P|\hat{\rho}_t|P\rangle = \frac{\exp(-[P - \langle \hat{P}(\phi)\rangle_t]^2/\{2\langle[\Delta \hat{P}(\phi)]^2\rangle_t\})}{\sqrt{2\pi\langle[\Delta \hat{P}(\phi)]^2\rangle_t}}, \quad (39)$$

where the dependence on F is in $\langle \hat{P}(\phi)\rangle$. Then $\mathcal{P}(\bar{P}|F)$ is also a Gaussian distribution with a variance equal to the sum of the two variances:

$$\mathcal{P}(\bar{P}|F) = \frac{\exp\left(-[\bar{P}-\langle\hat{P}(\phi)\rangle_t]^2\big/\left\{2\sigma_P^2+2\langle[\Delta\hat{P}(\phi)]^2\rangle_t\right\}\right)}{\sqrt{2\pi\left\{\sigma_P^2+\langle[\Delta\hat{P}(\phi)]^2\rangle_t\right\}}}. \quad (40)$$

The Fisher information corresponding to this measurement, obtained from Eq. (2), is

$$\mathcal{F}_P(F) = \frac{|D(t)|^2}{\sigma_P^2 + \langle[\Delta \hat{P}(\phi)]^2\rangle_t}. \quad (41)$$

The effect of the unsharp measurement is to lower the Fisher information by replacing the original final variance in the quadrature distribution of the oscillator by an effective variance, which includes the extra contribution due to the unsharp measurement.

Expressing $\langle[\Delta \hat{P}(\phi)]^2\rangle_t$ in Eq. (41) as in Eq. (12), one gets

$$\mathcal{F}_P(F) = \frac{|D(t)|^2}{\eta\left\{\langle[\Delta\hat{P}(\phi)]^2\rangle_0 + \sigma_P^2\right\} + (1-\eta)\left(n_T+\sigma_P^2+\frac{1}{2}\right)}, \quad (42)$$

which shows that unsharp measurements effectively increase the variance of the initial state of the oscillator and the temperature of the environment. Equation (36) is modified in the same way, and Eq. (32) changes to

$$\tilde{\mathcal{E}} = \frac{n_T + \sigma_P^2 + \frac{1}{2}}{[4(E+\sqrt{E^2-1/4})]^{-1} + \sigma_P^2}. \quad (43)$$

VII. APPLICATIONS

The uncertainty bound in Eq. (35) can be obtained by calculating $\overline{\omega^2}$ and ξ. For concreteness, we consider a force that has the form $\zeta(t) = \cos(\omega_F t)$ in the interior of the time interval t_0 and t_f, with the added proviso that it should be continuous and vanish smoothly at the extremes of this interval. We assume in the following that the time interval Δt of variation of $\zeta(t)$ around t_0 and t_f satisfies the inequalities $\gamma t_{\rm tot} \gg \gamma\Delta t \gg 8/\gamma\xi$, which implies that the contributions from the regions around the extremes of the interval are negligible, as can be seen from Eq. (D7). Two limiting time-dependencies of the applied force are especially interesting: the broadband regime $\omega_F t_{\rm tot} \ll 1$, which is effectively equivalent to constant forces, and the narrow-band regime $\omega_F t_{\rm tot} \gg 1$, which is effectively equivalent to harmonic forces. In both cases, we assume that $\omega t_{\rm tot} \gg 1$.

A. Broadband regime

Since this regime is, for the model here considered, equivalent to a constant force, we must have $\overline{\omega^2} \ll \omega^2$ and $\xi \simeq t_{\rm tot}$ so that from Eqs. (33) and (35) the optimal time and bound uncertainty become

$$\gamma\tau_{\rm opt} \simeq \left[\left(1+4\frac{\omega^2}{\gamma^2}\right)\frac{\mathcal{E}}{24}\right]^{-1/3}, \quad (44)$$

and

$$\delta f_{\mathcal{E}} \simeq \delta f_{\rm min}^{(BB)}\left\{1+\frac{1}{8}\left[1+\frac{4\omega^2}{\gamma^2}\right]^{1/3}\left(\frac{3}{\mathcal{E}}\right)^{2/3}\right\}, \quad (45)$$

with

$$\delta f_{\rm min}^{(BB)} = \sqrt{\frac{m\hbar\omega\gamma(n_T+1/2)}{2t_{\rm tot}}}. \quad (46)$$

Figure 2 displays the ratio $\delta f_{\mathcal{E}}/\delta f_{\rm min}^{(BB)}$ obtained by numerical optimization of ν for $\omega/\gamma = 100$ and $\omega t_{\rm tot} = 400\pi$, as a function of $\mathcal{E}$ (full black line). The expression in Eq. (45) corresponds to the red dotted line. Note that Eqs. (44) and (45) can be derived in a simple way, since for a constant force the coefficient $|D[t_0+n\tau; t_0+(n+1)\tau]|^2$ depends only on τ. This simpler derivation is shown in Appendix E.

B. Narrow-band regime

For narrow-band resonant forces ($\omega_F = \omega$), and for high energy $\mathcal{E} \gg 1$, two regimes emerge, depending on whether

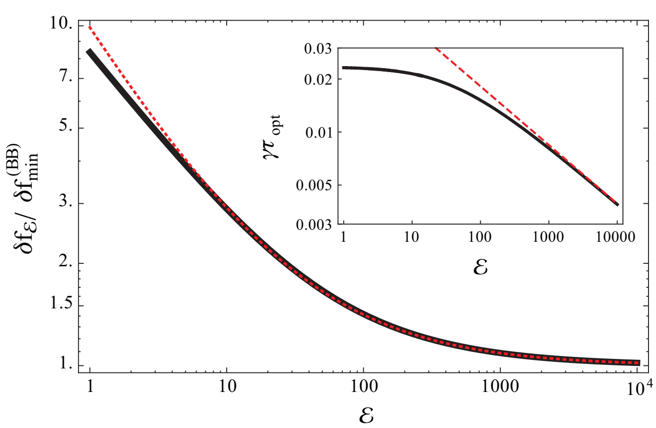

FIG. 2. (Color online) Ratio $\delta f_{\mathcal{E}}/\delta f_{\rm min}^{(BB)}$ for a constant force, applied during the time interval $t_{\rm tot} = 400\pi/\omega$, as a function of $\mathcal{E}$, for $\omega/\gamma = 100$. The full (black) curve is obtained by numerical optimization of ν, and the dotted (red) curve corresponds to the analytical approximation (45). The inset displays as a full (black) curve the optimal measurement time in units of $1/\gamma$ as a function of $\mathcal{E}$, while the dashed (red) curve corresponds to the analytical approximation (44).

$\mathcal{E} \gtrsim \omega/\gamma$ or $\mathcal{E} \lesssim \omega/\gamma$. One should note that this last condition is allowed only for weak-dissipation $\omega/\gamma \gg 1$. In the first regime, Eq. (35) applies, with $\overline{\omega^2} \simeq \omega^2$ and $\xi \simeq t_{\rm tot}/2$, leading to

$$\gamma\tau_{\rm opt} \simeq \left[\left(1+8\frac{\omega^2}{\gamma^2}\right)\frac{\mathcal{E}}{24}\right]^{-1/3}, \tag{47}$$

and

$$\delta f_{\mathcal{E}} \simeq \delta f_{\rm min}^{(NB)}\left\{1+\frac{1}{8}\left[1+\frac{8\omega^2}{\gamma^2}\right]^{1/3}\left(\frac{3}{\mathcal{E}}\right)^{2/3}\right\}, \tag{48}$$

with

$$\delta f_{\rm min}^{(NB)} = \sqrt{\frac{m\hbar\omega\gamma(n_T+1/2)}{t_{\rm tot}}}. \tag{49}$$

In the second regime, Eq. (35) does not hold any more. In this case, however, the factor $|D(t,t+\tau)|$ can be simplified by using the rotating-wave approximation (RWA):

$$\begin{aligned}|D(t,t+\tau)| &= \omega\left|\int_t^{t+\tau} dt'\zeta(t')e^{i\omega t'}e^{-\gamma(t+\tau-t')/2}\right| \\ &= \frac{\omega}{2}\left|\frac{2}{\gamma}(1-e^{-\gamma\tau/2})e^{2i\omega t}\frac{e^{2\omega i\tau}-e^{-\gamma\tau/2}}{2\omega i+\gamma/2}\right| \\ &\simeq \frac{\omega}{\gamma}(1-e^{-\gamma\tau/2}) \equiv D_{\rm RWA}(\tau),\end{aligned} \tag{50}$$

where we have taken $\zeta(t) = \cos(\omega t)$, consistent with the narrow-band condition (note that this function should be smoothed at t_0 and t_f, so as to satisfy the condition of continuity). The last step in Eq. (50), which corresponds to neglecting fast-oscillating terms, is valid when

$$(1-e^{-\gamma\tau/2}) \gg \left|\frac{e^{-2i\omega\tau}-e^{-\gamma\tau/2}}{1+4i\omega/\gamma}\right|. \tag{51}$$

This condition is equivalent to

$$4\frac{\omega}{\gamma}\tanh\left(\frac{\gamma\tau}{4}\right) \gg 1. \tag{52}$$

This inequality expresses the region of validity of the RWA, in the presence of dissipation. It implies that $\omega\tau \gg 1$. This means that, in order for this procedure to be valid, the optimal time should satisfy this condition. This is verified to be indeed the case in the following.

Since $D_{\rm RWA}(\tau)$ does not depend on t, the sum in Eq. (31) reduces to $\sum_{n=0}^{\nu-1}|D[t_0+n\tau;t_0+(n+1)\tau]|^2 = \nu|D_{\rm RWA}(\tau)|^2$, simplifying the quantum Fisher information to a function of τ only. The maximization over τ leads to the optimal time $\tau_{\rm opt} \simeq (24/\mathcal{E})^{1/3}/\gamma$ (see Appendix E). For $\mathcal{E} \lesssim \omega/\gamma$, one has $\omega\tau_{\rm opt} \gg 1$, which justifies the use of the RWA. The corresponding bound for the uncertainty is

$$\delta f_{\mathcal{E}} \simeq \sqrt{2}\delta f_{\rm min}^{(NB)}[1+(1/8)(3/\mathcal{E})^{2/3}]. \tag{53}$$

This implies that, for a narrow-band resonance force, the achievable precision in the regime of validity of the RWA is worse only by a factor $\sqrt{2}$ with respect to Eq. (49).

Figure 3 displays the behavior of $\mathcal{F}_Q(t_{\rm tot}/\nu,\mathcal{E})/(n_T+1/2)$ as a function of ν, for $\omega/\gamma = 100$, $t_{\rm tot} = 200(2\pi/\omega)$ and for

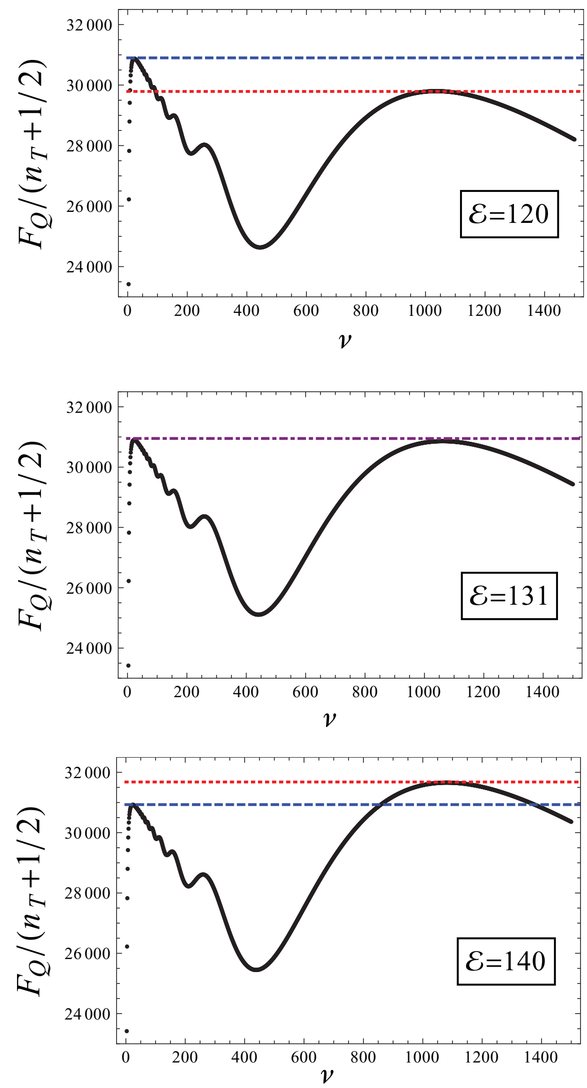

FIG. 3. (Color online) $\mathcal{F}_Q(t_{\rm tot}/\nu,\mathcal{E})/(n_T+1/2)$ as a function of the number of measurements ν, for $\omega/\gamma = 100$, $t_{\rm tot} = 200(2\pi/\omega)$ and for $\mathcal{E} = 120$ (top panel), $\mathcal{E} = 131$ (central panel), and $\mathcal{E} = 140$ (bottom panel). The dashed (blue) lines correspond to the first local maximum, and the dotted (red) ones correspond to the last local maximum. For $\mathcal{E} = 131$, the dot-dashed (purple) line corresponds to the two local maxima, which coincide for this value of $\mathcal{E}$.

$\mathcal{E} = 120$ (top panel), $\mathcal{E} = 131$ (central panel), and $\mathcal{E} = 140$ (bottom panel). The number ν of measurements that leads to a maximization of $\mathcal{F}_Q^S(t_{\rm tot}/\nu)$ depends on the value of $\mathcal{E}$. This optimal number is denoted $\nu_{\rm opt}$, so that $\tau_{\rm opt}$ corresponds to $t_{\rm tot}/\nu_{\rm opt}$. For $\mathcal{E} \simeq 131$, there are two global maxima, corresponding to the values 22 and 1050 of ν.

Figure 4 displays the ratio $\delta f_{\mathcal{E}}/\delta f_{\rm min}^{(NB)}$, obtained by numerical optimization of ν, for $\omega/\gamma = 100$ and $\omega t_{\rm tot} = 400\pi$, as a function of $\mathcal{E}$ (full black line). The expressions in Eqs. (48) and (53) correspond to the red dotted and purple dashed lines, respectively. The transition between the two asymptotic regimes (48) and (53) leads to the discontinuity at $\mathcal{E} = \mathcal{E}_{\rm trans} \simeq 131$ in the derivative displayed in Fig. 4. The inset exhibits the corresponding discontinuity of $\tau_{\rm opt}$, occurring

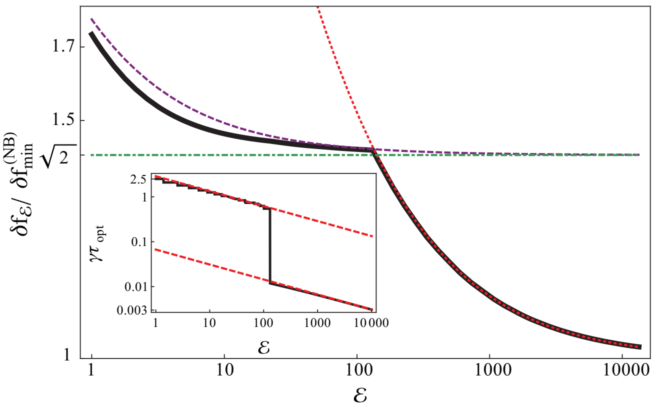

FIG. 4. (Color online) Ratio $\delta f_\mathcal{E}/\delta f_{\min}^{(NB)}$ as a function of $\mathcal{E}$ for $\omega/\gamma = 100$ and $\omega t_{\rm tot} = 400\pi$. The full (black) curve is obtained by numerical optimization of ν, the dotted (red) and the dashed (purple) curves are obtained from the analytical expansions for, respectively, (i) high-$\mathcal{E}$ regime ($\mathcal{E} \gtrsim \omega/\gamma$) and (ii) the regime where the RWA applies ($\mathcal{E} \lesssim \omega/\gamma$ and $\mathcal{E} \gg 1$). The dashed-dotted (green) line displays the ultimate limit for the force uncertainty in the RWA. The transition between the two regimes is characterized by the discontinuity of the derivative of $\delta f_\mathcal{E}$ as a function of $\mathcal{E}$. The full (black) curve in the inset, obtained numerically, displays the discontinuity of $\tau_{\rm opt}$ that leads to this transition. The upper and lower dashed (red) lines correspond respectively to the regimes (i) and (ii).

for the same value of $\mathcal{E}$. These discontinuities can be explained by the appearance of the two global maxima in Fig. 3.

Figure 5 compares the optimal times of measurement M_1, M_2, and M_3 with the modulation of the classical force, for two values of $\mathcal{E}$, one ($\mathcal{E} = 60$, panel *a*) below and the other ($\mathcal{E} = 200$, panel *b*) beyond the discontinuity.

We may estimate the value of $\mathcal{E}_{\rm trans}$ by looking for the points where the asymptotes for RWA [Eq. (53)] and without RWA [Eq. (48)] meet, which yields

$$\mathcal{E}_{\rm trans} \simeq 3\left[\frac{(1+8\omega^2/\gamma^2)^{1/3} - \sqrt{2}}{8(\sqrt{2}-1)}\right]^{3/2}. \quad (54)$$

For $\omega/\gamma = 100$, this leads to $\mathcal{E}_{\rm trans} = 134$, which should be compared to 131, the numerical value for $\mathcal{E}$ at the transition point.

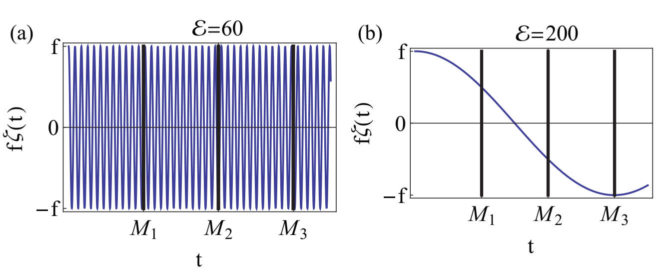

FIG. 5. (Color online) Figures *a* and *b* compare the instants of time corresponding to the first three measurements, denoted by M_1, M_2, and M_3, and separated by the optimal time $\tau_{\rm opt}$, with the modulation of the classical force, for two values of $\mathcal{E}$, one [$\mathcal{E} = 60$, panel (a)] below and the other [$\mathcal{E} = 200$, panel (b)] beyond the discontinuity.

If the oscillator is initially in the ground state, then, from Eq. (32), $\mathcal{E} = 2n_T + 1$, so that $\mathcal{E}$ can be thought as a measure of the temperature of the environment. In this case, Fig. 3 shows that the temperature, besides affecting the precision, determines the best time interval in the sequential measurement. Equation (54) leads then to an approximate expression for the transition temperature $T_{\rm trans}$, such that if $T < T_{\rm trans}$ the optimal time is much larger than the period of the oscillator.

C. Application to trapped ions

Equation (24) can be directly applied to the measurement of resonant forces acting on trapped ions, where diffusive noise plays an important role [10]. This corresponds to the double limit $\gamma \to 0$, $n_T \to \infty$, with $\gamma n_T = \mathcal{D}$ (constant). In the narrow-band regime, one gets then from (24) that, for a single measurement at the time $t_{\rm tot}$, as considered in Ref. [10],

$$\delta f \geqslant \sqrt{4m\hbar\omega\mathcal{D}/t_{\rm tot}}\sqrt{1 + \{4\mathcal{D}\langle[\Delta\hat{X}(\phi_{t_{\rm tot}})]^2\rangle_0 t_{\rm tot}\}^{-1}}. \quad (55)$$

The first factor on the right-hand side of the above equation corresponds to the expression derived through heuristic arguments in Ref. [10]. It is seen to overestimate the reachable precision for this measurement strategy. This is due to the fact that the quantum noise in the initial state of the oscillator was neglected. This can only be done when the diffusion noise surpasses the quantum one; that is, when $4\mathcal{D}\langle[\Delta\hat{X}(\phi_{t_{\rm tot}})]^2\rangle_0 t_{\rm tot} \gg 1$. For the conditions assumed in Ref. [10], this corresponds to $t_{\rm tot} \gg 1$ ms. Under these conditions, the lower bound for δf is larger by a factor $\sqrt{2}$ than the bound given by Eq. (36), which corresponds to the sequential measurement strategy described in Sec. V.

VIII. CONCLUSION

In conclusion, we have been able to completely solve a precision problem involving the estimation of the amplitude of a general time-dependent classical force through measurements on a probe consisting of a noisy quantum-mechanical oscillator. The force is estimated through a discrete sequence of measurements on the oscillator at optimal time intervals. We have determined the ultimate precision limit, as a function of the average energy of the oscillator and the temperature of the environment, and also the best probe state and the best measurement procedure, thus yielding a rigorous and useful benchmark for experiments that aim to detect extremely small forces and displacements.

ACKNOWLEDGMENTS

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APPENDIX A: UNITARY EVOLUTION DESCRIPTION FOR FORCED HARMONIC OSCILLATOR AT ZERO TEMPERATURE

We show here that the unitary evolution described in Sec. IV is equivalent to the master equation for the forced noisy harmonic oscillator. To this end, we demonstrate first that the simultaneous actions of the external force and the noise can be decomposed into two successive operations. The first one is a purely dissipative evolution, corresponding to an interaction of the system with a zero-temperature thermal reservoir, and the second one corresponds to a displacement in phase-space, attenuated with respect to the displacement of the noiseless case.

In the Markov limit, the master equation that describes the evolution of a harmonic oscillator driven by the force defined above, and in contact with a zero-temperature reservoir, in the interaction picture, is given by

$$\frac{d\hat{\rho}(t)}{dt} = iF\{\alpha(t)[\hat{a},\hat{\rho}(t)] + \alpha^*(t)[\hat{a}^\dagger,\hat{\rho}(t)]\} - \frac{\gamma}{2}[\hat{a}^\dagger\hat{a}\hat{\rho}(t) + \hat{\rho}(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}(t)\hat{a}^\dagger]. \quad \text{(A1)}$$

This master equation may be simplified with a more convenient picture. Defining the density operator of the oscillator in this picture by

$$\hat{\rho}_L(t) = \hat{U}_L^\dagger(t)\hat{\rho}(t)\hat{U}_L(t), \quad \text{(A2)}$$

where

$$\hat{U}_L(t) = e^{iF[L(t)\hat{a}+L^*(t)\hat{a}^\dagger]} \quad \text{(A3)}$$

and

$$L(t) = \int_0^t dt'\alpha(t')e^{-\frac{\gamma}{2}(t-t')}, \quad \text{(A4)}$$

with $\alpha(t)$ defined by Eq. (7). This implies that

$$\hat{U}_L(t) = e^{iF|D(t)|\hat{X}(\phi_t)}, \quad \text{(A5)}$$

with $|D(t)| = \sqrt{2}|L(t)|$, $\hat{X}(\phi_t) = (\hat{a}e^{-i\phi_t} + \hat{a}^\dagger e^{i\phi_t})/\sqrt{2}$, and $\phi_t = -\arg L(t)$. The density operator $\hat{\rho}_L(t)$ satisfies then the equation

$$\frac{d\hat{\rho}_L(t)}{dt} = -\frac{\gamma}{2}[\hat{a}^\dagger\hat{a}\hat{\rho}_L(t) + \hat{\rho}_L(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_L(t)\hat{a}^\dagger], \quad \text{(A6)}$$

which only contains the effects of the interaction between the oscillator and the zero-temperature reservoir.

Therefore, in the presence of the coupling with a thermal reservoir, the displacement in phase space due to the applied force is modified from the lossless case only by a damping factor $e^{-\gamma(t-t')/2}$ inside the time integral. This corresponds to the expression for $D(t)$ in Eq. (10), obtained in Sec. III from the Langevin equation.

We show now that there is a convenient purification of the nonunitary evolution stemming from the above master equation, which yields a tight bound for the quantum Fisher information corresponding to the estimation of the force acting on the harmonic oscillator. This purification is used in Sec. IV.

The solution of (A6) may be found, for instance, in Ref. [49]. Going back to the interaction picture, one can see that the harmonic oscillator will evolve as

$$\hat{\rho}(t) = \hat{U}_L(t)\left[\sum_{n=0}^{\infty}\frac{(1-\eta)^n}{n!}\eta^{\hat{a}^\dagger\hat{a}/2}\hat{a}^n\hat{\rho}_0(\hat{a}^\dagger)^n\eta^{\hat{a}^\dagger\hat{a}/2}\right]\hat{U}_L^\dagger(t), \quad \text{(A7)}$$

where $\hat{\rho}_0 = |\psi_0\rangle_{S\,S}\langle\psi_0|$, and $|\psi_0\rangle_S$ is the initial state of system S.

This nonunitary evolution can be seen as resulting from a unitary evolution on an enlarged Hilbert space comprising the system and an environment, when this environment is not monitored. Here, we represent the environment by another harmonic oscillator and show that the unitary interaction between the system and this environment leads to the nonunitary evolution given in Eq. (A7).

Let then

$$|\Psi(t)\rangle = \hat{U}_L(t)\sum_{n=0}^{\infty}\sqrt{\frac{(1-\eta)^n}{n!}}\eta^{\hat{a}^\dagger\hat{a}/2}\hat{a}^n|\psi_0\rangle_S|n\rangle_R = \hat{U}_L(t)\sum_{n=0}^{\infty}\frac{(1-\eta)^{n/2}}{n!}\eta^{\hat{a}^\dagger\hat{a}/2}\hat{a}^n(\hat{b}^\dagger)^n|\psi_0\rangle_S|0\rangle_R \quad \text{(A8)}$$

be a purification of $\hat{\rho}(t)$; that is, if one traces out the environment in $|\Psi(t)\rangle\langle\Psi(t)|$, one is left with $\hat{\rho}(t)$. In the above equation, $|n\rangle_R$ are Fock states of the environment R, and $\hat{a}$ ($\hat{b}$) is an annihilation operator corresponding to system S (environment R). Now, it is straightforward to show that $|\Psi(t)\rangle$ can be rewritten as

$$|\Psi(t)\rangle = e^{iF|D(t)|\hat{X}(\phi_t)}\hat{B}_1|\psi_0\rangle_S|0\rangle_R, \quad \text{(A9)}$$

where $\hat{B}_1 = e^{\arccos(\sqrt{\eta})(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})}$ can be seen as the transformation performed by a beam splitter with transmissivity η on the input modes represented by $\hat{a}$ and $\hat{b}$. Since $|\Psi(t)\rangle$ is a purification of $\hat{\rho}(t)$, the evolution described by the master equation (A1) can equivalently be modeled by a beam-splitter-like unitary interaction between the system S and the environment R (represented here by a harmonic oscillator), followed by a displacement in phase space of the system S, when the environment R is not monitored (is traced out).

APPENDIX B: UNITARY EVOLUTION DESCRIPTION FOR FORCED HARMONIC OSCILLATOR UNDER THERMAL RESERVOIR WITH ARBITRARY TEMPERATURE

In this Appendix, we generalize the results in Appendix A to nonzero temperatures. As before, the simultaneous actions of the external force and the thermal noise can be decomposed into two successive operations.

The corresponding master equation, in the interaction picture, is

$$\frac{d\hat{\rho}(t)}{dt} = iF\{\alpha(t)[\hat{a},\hat{\rho}(t)] + \alpha^*(t)[\hat{a}^\dagger,\hat{\rho}(t)]\} - \frac{\gamma}{2}(n_T+1)[\hat{a}^\dagger\hat{a}\hat{\rho}(t) + \hat{\rho}(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}(t)\hat{a}^\dagger] - \frac{\gamma n_T}{2}[\hat{a}\hat{a}^\dagger\hat{\rho}(t) + \hat{\rho}(t)\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{\rho}(t)\hat{a}]. \quad \text{(B1)}$$

If one defines

$$\hat{\rho}_L(t) = e^{-iF[L(t)\hat{a}+L^*(t)\hat{a}^\dagger]}\hat{\rho}(t)e^{iF[L(t)\hat{a}+L^*(t)\hat{a}^\dagger]},$$

with $L(t)$ defined in (A4), then $\hat{\rho}_L$ satisfies the above master equation without the force term.

The solution of (B1) without the force term is [49]

$$\hat{\rho}_L(t) = r_3 e^{\ln[r_2]\hat{a}^\dagger\hat{a}} \sum_{l,j=0}^{\infty} \left[\frac{(n_T+1)^l (n_T)^j r_1^{l+j}}{l!j!r_2^{2j}} \times (\hat{a}^\dagger)^j \hat{a}^l \hat{\rho}_0 (\hat{a}^\dagger)^l \hat{a}^j \right] e^{\ln[r_2]\hat{a}^\dagger\hat{a}}, \tag{B2}$$

where the functions r_1, r_2, and r_3 are defined by

$$r_1 = \frac{1-\eta}{[n_T(1-\eta)+1]}, \tag{B3}$$

$$r_2 = \frac{\sqrt{\eta}}{n_T(1-\eta)+1}, \tag{B4}$$

$$r_3 = \frac{1}{n_T(1-\eta)+1}, \tag{B5}$$

and $\eta = e^{-\gamma t}$.

Therefore, in the interaction picture, the evolution of $\hat{\rho}(t)$ is

$$\hat{\rho}(t) = e^{iF|D(t)|\hat{X}(\phi_t)} \left\{ r_3 e^{\ln[r_2]\hat{a}^\dagger\hat{a}} \sum_{l,j=0}^{\infty} \left[\frac{(n_T+1)^l (n_T)^j r_1^{l+j}}{l!j!r_2^{2j}} (\hat{a}^\dagger)^j \hat{a}^l \hat{\rho}_0 (\hat{a}^\dagger)^l \hat{a}^j \right] e^{\ln[r_2]\hat{a}^\dagger\hat{a}} \right\} e^{-iF|D(t)|\hat{X}(\phi_t)}. \tag{B6}$$

A purification of $\hat{\rho}(t)$ can be built with an environment consisting of two harmonic oscillators:

$$\begin{aligned} |\Psi(t)\rangle &= e^{iF|D(t)|\hat{X}(\phi_t)} e^{\ln[r_2]\hat{a}^\dagger\hat{a}} \sum_{l,j=0}^{\infty} \sqrt{\frac{(n_T+1)^l (n_T)^j r_1^{l+j} r_3}{l!j!r_2^{2j}}} (\hat{a}^\dagger)^j \hat{a}^l |\psi_0\rangle_S |l\rangle_{R_1} |j\rangle_{R_2} \\ &= e^{iF|D(t)|\hat{X}(\phi_t)} e^{\ln[r_2]\hat{a}^\dagger\hat{a}} \sum_{l,j=0}^{\infty} \sqrt{\frac{(n_T+1)^l (n_T)^j r_1^{l+j} r_3}{r_2^{2j}}} \frac{(\hat{a}^\dagger\hat{c}^\dagger)^j}{j!} \frac{(\hat{a}\hat{b}^\dagger)^l}{l!} |\psi_0\rangle_S |0\rangle_{R_1} |0\rangle_{R_2}, \end{aligned} \tag{B7}$$

where $|l\rangle_{R_1}$ and $|j\rangle_{R_2}$ are Fock states of the environments R_1 and R_2 respectively, and $\hat{b}$ ($\hat{c}$) is the annihilation operator for the environment R_1 (R_2). This purification involves three unitary evolutions: the first one corresponds to a beam-splitter-like interaction between the system S and the environment R_1, the second one corresponds to a two-mode squeezing-like interaction between the system S and the environment R_2, and the third one corresponds to a phase-space displacement in S space. In fact, the above purification may be rewritten as

$$|\Psi(t)\rangle = e^{iF|D(t)|\hat{X}(\phi_t)} \hat{S}\hat{B}_1 |\psi_0\rangle_S |0\rangle_{R_1} |0\rangle_{R_2}, \tag{B8}$$

where both environments are taken initially in the ground state, and

$$\hat{B}_1 = e^{\theta_1(t)(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})}, \tag{B9}$$

$$\hat{S} = e^{\theta_2(t)(\hat{a}^\dagger\hat{c}^\dagger - \hat{a}\hat{c})}, \tag{B10}$$

with $\theta_1(t)$ and $\theta_2(t)$ given by

$$\theta_1(t) = \arccos\left[\sqrt{\frac{\eta}{n_T(1-\eta)+1}}\right], \tag{B11}$$

$$\theta_2(t) = \text{arccosh}[\sqrt{n_T(1-\eta)+1}]. \tag{B12}$$

When $T=0$, it follows from these expressions that $\theta_2(t)=0$, implying that $\hat{S}=\hat{1}$, so that the harmonic oscillator R_2 is decoupled from the other two oscillators and becomes thus superfluous. One recovers then the description in Sec. IV, corresponding to Fig. 1.

APPENDIX C: MINIMIZATION OF QUANTUM FISHER INFORMATION OF SYSTEM PLUS ENVIRONMENT AT ARBITRARY TEMPERATURE

Any two purifications of a density operator can be related by a unitary transformation acting on the environment alone [45]. Therefore, the most general purification of (B6) is given by

$$|\Phi\rangle = e^{iF|D(t)|\hat{H}_{1,2}} |\Psi(t)\rangle, \tag{C1}$$

where $\hat{H}_{1,2}$ is a Hermitian operator acting only on the environments R_1 and R_2, and $|\Psi(t)\rangle$ is defined in Eq. (B7). In order to get the lowest upper bound for the quantum Fisher information of the system, the operator $\hat{H}_{1,2}$ should be chosen properly in order to minimize the quantum Fisher information corresponding to $S+R_1+R_2$.

For a given $\hat{H}_{1,2}$, an upper bound to the quantum Fisher information corresponding to S may be calculated from $|\Phi\rangle$, yielding

$$\begin{aligned} \mathcal{F}_Q^{SR_1R_2} = [2|D(t)|]^2_{R_1} & \langle 0|_{R_2} \langle 0|_S \langle \psi_0| \\ & \times (\Delta\{\hat{B}_1^\dagger \hat{S}^\dagger [\hat{X}_S(\phi_t) + \hat{H}_{1,2}] \hat{S}\hat{B}_1\})^2 \\ & \times |\psi_0\rangle_S |0\rangle_{R_2} |0\rangle_{R_1}. \end{aligned} \tag{C2}$$

As discussed in Sec. IV, a possible choice of the operator $\hat{H}_{1,2}$ is $\hat{H}_{1,2} = \lambda_1 \hat{X}_{R_1}(\phi_t) + \lambda_2 \hat{X}_{R_2}(\phi_t)$, where $\hat{X}_{R_1}(\phi_t)$ [$\hat{X}_{R_2}(\phi_t)$] is the rotated quadrature operator of the oscillator in R_1 (R_2) space. For $\hat{H}_{1,2}=0$, after disentangling $S+R_1+R_2$ with the operation $\hat{B}_1^\dagger \hat{S}^\dagger$, which does not change the quantum

Fisher information, the effective unitary evolution in $S + R_1 + R_2$ is

$$\hat{U}_{S,R_1,R_2} = \exp\big(iF|D(t)|\{\cosh[\theta_2(t)]\cos[\theta_1(t)]\,\hat{X}_S(\phi_t) - \cosh[\theta_2(t)]\sin[\theta_1(t)]\,\hat{X}_{R_1}(\phi_t) + \sinh[\theta_2(t)]\,\hat{X}_{R_2}(\phi_t)\}\big). \tag{C3}$$

It is clear that, with $\hat{H}_{1,2} = \lambda_1\hat{X}_{R_1}(\phi_t) + \lambda_2\hat{X}_{R_2}(\phi_t)$ and convenient values of λ_1 and λ_2, it is possible to erase at least part of the nonredundant information in $|\Psi(t)\rangle$.

After a straightforward calculation, Eq. (C2), with the above choice of $\hat{H}_{1,2}$, can be rewritten as

$$\begin{aligned}\frac{\mathcal{F}_Q^{SR_1R_2}(\lambda_1,\lambda_2)}{[2|D(t)|]^2} &= \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0(\{\cosh[\theta_2(t)] + \lambda_2\sinh[\theta_2(t)]\}\cos[\theta_1(t)] + \lambda_1\sin[\theta_1(t)])^2\\ &+\big\langle\big[\Delta\hat{X}_{R_1}(\phi_t)\big]^2\big\rangle_0(-\{\cosh[\theta_2(t)] + \lambda_2\sinh[\theta_2(t)]\}\sin[\theta_1(t)] + \lambda_1\cos[\theta_1(t)])^2\\ &+\big\langle\big[\Delta\hat{X}_{R_2}(\phi_t)\big]^2\big\rangle_0(\sinh[\theta_2(t)] + \lambda_2\cosh[\theta_2(t)])^2.\end{aligned} \tag{C4}$$

Then, the optimal values of λ_1 and λ_2 that minimize the above equation are

$$\lambda_1^{(\mathrm{opt})} = \frac{\big\{\frac{1}{2} - \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0\big\}\big\{\frac{1}{2}\cos[\theta_1(t)] - \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0\big\}\cos[\theta_1(t)]\sin[\theta_1(t)]\cosh^{-1}[\theta_2(t)]}{\big\{\frac{1}{2}\cos^2[\theta_1(t)] + \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0\sin^2[\theta_1(t)]\big\}\big(\frac{1}{2}\cos^2[\theta_1(t)] + \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0\{\sin^2[\theta_1(t)] + \tanh[\theta_2(t)]\}\big)},$$

$$\lambda_2^{(\mathrm{opt})} = -\tanh[\theta_2(t)]\left[\frac{\frac{1}{2}\cos^2[\theta_1(t)] + \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0\{1 + \sin^2[\theta_1(t)]\}}{\frac{1}{2}\cos^2[\theta_1(t)] + \langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0\{\sin^2[\theta_1(t)] + \tanh^2[\theta_2(t)]\}}\right],$$

where we have used that $\langle[\Delta\hat{X}_{R_1}(\phi_t)]^2\rangle_0 = \langle[\Delta\hat{X}_{R_2}(\phi_t)]^2\rangle_0 = 1/2$. Therefore, the minimum value of $\mathcal{F}_Q^{SR_1R_2}(\lambda_1,\lambda_2)$ is given by

$$\frac{\mathcal{F}_Q^{SR_1R_2}\big(\lambda_1^{(\mathrm{opt})},\lambda_2^{(\mathrm{opt})}\big)}{[2|D(t)|]^2} = \left\{\frac{\sinh^2[\theta_2(t)]}{\big\langle\big[\Delta\hat{X}_{R_1}(\phi_t)\big]^2\big\rangle_0} + 2\cosh^2[\theta_2(t)]\sin^2[\theta_1(t)] + 2\cosh^2[\theta_2(t)]\cos^2[\theta_1(t)]\right\}^{-1}. \tag{C5}$$

Substituting $\theta_1(t)$ and $\theta_2(t)$ in terms of η and n_T, we get the upper bound

$$\mathcal{F}_Q^{SR_1R_2}\big(\lambda_1^{(\mathrm{opt})},\lambda_2^{(\mathrm{opt})}\big) = [2|D(t)|]^2\left\{2(1-\eta)(2n_T+1) + \frac{\eta}{\langle[\Delta\hat{X}_S(\phi_t)]^2\rangle_0}\right\}^{-1}. \tag{C6}$$

APPENDIX D: SEQUENTIAL-MEASUREMENT PROCEDURE

Here the quantum Fisher information in Eq. (30) is maximized by choosing optimal time intervals τ.

In the limit of a rapid sequential measurement, that is, τ is small compared to the characteristic time evolution of $|D(t;t+\tau)|^2$, the following expansion holds:

$$\begin{aligned}|D(t;t+\tau)|^2 &= \omega^2 e^{-\gamma(t+\tau)}\left|\int_t^{t+\tau} dt'\zeta(t')e^{t'(\gamma/2+i\omega)}\right|^2\\ &= \omega^2 e^{-\gamma(t+\tau)}\left|\frac{\tau}{1!}\zeta(t)e^{t(\gamma/2+i\omega)} + \frac{\tau^2}{2!}\frac{d}{dt}[\zeta(t)e^{t(\gamma/2+i\omega)}] + \frac{\tau^3}{3!}\frac{d^2}{dt^2}[\zeta(t)e^{t(\gamma/2+i\omega)}] + O(\tau^4)\right|^2\\ &= \omega^2 e^{-\gamma\tau}[A_1(t)\tau^2 + A_2(t)\tau^3 + A_3(t)\tau^4 + O(\tau^5)],\end{aligned} \tag{D1}$$

in which

$$A_1(t) = \zeta(t)^2, \tag{D2}$$

$$A_2(t) = \frac{\gamma}{2}\zeta^2(t) + \zeta'(t)\zeta(t), \tag{D3}$$

$$A_3(t) = \frac{1}{3}\zeta''(t)\zeta(t) + \frac{1}{4}\zeta'^2(t) + \frac{7\gamma}{12}\zeta'(t)\zeta(t) + \frac{1}{12}\left(\frac{7}{4}\gamma^2 - \omega^2\right)\zeta^2(t), \tag{D4}$$

where $\zeta'(t)$ and $\zeta''(t)$ are the first and the second time derivatives of $\zeta(t)$, respectively. Note that this expansion is formally justified when τ is much smaller than $1/\gamma$, $1/\omega$, and the characteristic time of evolution of $\zeta(t)$. Under these conditions, we have

$$\mathcal{F}_Q(\tau,\mathcal{E}) = \frac{2\omega^2/(2n_T+1)}{e^{\gamma\tau} - 1 + 1/\mathcal{E}}\sum_{n=0}^{\nu-1}[A_1(t_0+n\tau)\tau^2 + A_2(t_0+n\tau)\tau^3 + A_3(t_0+n\tau)\tau^4 + O(\tau^5)].$$

Now to maximize $\mathcal{F}_Q(\tau,\mathcal{E})$ as a function of τ, we use the Euler-Maclaurin formula, so that

$$\sum_{n=0}^{\nu-1} A_i(t_0+n\tau)\tau = \int_{t_0}^{t_f} dt A_i(t) - \frac{\tau}{2}[A_i(t_f)-A_i(t_0)] + \frac{\tau^2}{12}\left[\frac{dA_i}{dt}(t_f) - \frac{dA_i}{dt}(t_0)\right] + O(\tau^4).$$

The quantum Fisher information rewrites

$$\mathcal{F}_Q(\tau,\mathcal{E}) = \frac{2\omega^2/(2n_T+1)}{e^{\gamma\tau}-1+1/\mathcal{E}}\gamma^{-1}\xi(t_0;t_f) \times \left\{(\gamma\tau) + \frac{(\gamma\tau)^2}{2!} + \kappa(t_0;t_f)\frac{(\gamma\tau)^3}{3!} + O[(\gamma\tau)^4]\right\},$$

where

$$\xi(t_0;t_f) = \int_{t_0}^{t_f} dt\,\zeta^2(t), \tag{D5}$$

$$\kappa(t_0;t_f) = -\frac{1}{8}\left[\left(\frac{4\omega^2}{\gamma^2}-8\right) + \frac{\chi(t_0;t_f)}{\xi(t_0;t_f)}\right], \tag{D6}$$

and

$$\chi(t_0;t_f) = \int_{t_0}^{t_f} dt\left[\frac{2\zeta'(t)}{\gamma} - \zeta(t)\right]^2. \tag{D7}$$

The optimal interval $\tau_{\rm opt}$ that maximizes $\mathcal{F}_Q(\tau,\mathcal{E})$ is sufficiently large for the force to imprint measurable information on the harmonic oscillator but not too long so that the state of the harmonic oscillator be strongly disturbed by the environment. We show here that $\tau_{\rm opt}$ goes to zero as $\mathcal{E}$ goes to infinity. Equating to zero the derivative of $\mathcal{F}_Q(\tau,\mathcal{E})$ with respect to τ, and rearranging the terms, an approximation for $\tau_{\rm opt}$ is given implicitly by

$$\frac{1}{\mathcal{E}} \simeq 1 - e^{\gamma\tau_{\rm opt}} + e^{\gamma\tau_{\rm opt}} \times \frac{(\gamma\tau_{\rm opt}) + (\gamma\tau_{\rm opt})^2/2 + \kappa(t_0;t_f)(\gamma\tau_{\rm opt})^3/6}{1+(\gamma\tau_{\rm opt}) + \kappa(t_0;t_f)(\gamma\tau_{\rm opt})^2/2}, \tag{D8}$$

which is given by

$$\frac{1}{\mathcal{E}} = \frac{1}{3}(1-\kappa)(\gamma\tau_{\rm opt})^3 + O[(\gamma\tau_{\rm opt})^4]. \tag{D9}$$

So, to lowest order, the optimal time is given by

$$\gamma\tau_{\rm opt} = \left[(1-\kappa)\frac{\mathcal{E}}{3}\right]^{-1/3} = \left\{\frac{1}{24}\left[\frac{4\omega^2}{\gamma^2} + \frac{\chi(t_0;t_f)}{\xi(t_0;t_f)}\right]\mathcal{E}\right\}^{-1/3}. \tag{D10}$$

The functions ξ and χ may be expressed in the frequency domain. Let us assume that $\zeta(t)$ and $\zeta'(t)$ are continuous everywhere. We define the modulation of the classical force in the frequency domain as

$$\tilde{\zeta}(\omega') \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\,\zeta(t)e^{i\omega' t}. \tag{D11}$$

Notice that the limits of integration in Eqs. (D5) and (D7) can be taken from $-\infty$ to ∞, since the modulation $\zeta(t)$ is zero outside the interval $[t_0,t_f]$. We get then

$$\xi = \int_{-\infty}^{\infty} |\tilde{\zeta}(\omega')|^2 d\omega', \tag{D12}$$

$$\chi = \int_{-\infty}^{\infty} |\tilde{\zeta}(\omega')|^2\left(1+\frac{4\omega'^2}{\gamma^2}\right)d\omega', \tag{D13}$$

which shows that

$$\frac{\chi}{\xi} = 1 + 4\frac{\overline{\omega^2}}{\gamma^2}, \tag{D14}$$

where $\overline{\omega^2}$ is the mean-square frequency of the classical force, defined by Eq. (34).

From these considerations, we derive a bound $\delta f_{\mathcal{E}}$ for the uncertainty in the estimation of the force amplitude:

$$\delta f_{\mathcal{E}} \equiv \sqrt{\frac{m\hbar\omega^3}{\mathcal{F}_Q(\tau_{\rm opt},\mathcal{E})}} = \sqrt{\frac{m\hbar\omega\gamma(2n_T+1)}{2\xi}} \times \left\{1 + \frac{1}{8}\left[1+\frac{4(\omega^2+\overline{\omega^2})}{\gamma^2}\right]^{1/3}\left(\frac{3}{\mathcal{E}}\right)^{2/3} + O(\mathcal{E}^{-1})\right\}. \tag{D15}$$

Note that, for $\mathcal{E}$ going to infinity, the minimum uncertainty $\delta f_{\rm min}$, defined as

$$\delta f_{\rm min} \equiv \sqrt{\frac{m\hbar\omega\gamma(2n_T+1)}{2\xi}}, \tag{D16}$$

is achieved. This bound coincides with the one derived in Ref. [3] (where the decay constant was defined as $H = m\gamma/2$).

APPENDIX E: OPTIMAL TIME FOR CONSTANT FORCES AND FOR ROTATING-WAVE APPROXIMATION

If the applied force is constant during the time interval $t_{\rm tot}$, or if the rotating-wave approximation holds, $|D(t;t+\tau)|$ depends only on τ. Then, the sum that appears in Eq. (31),

$$\sum_{n=0}^{\nu-1} |D[t_0+n\tau;t_0+(n+1)\tau]|^2, \tag{E1}$$

simplifies to $\nu|D(\tau)|^2$, where $D(\tau)$ is given by Eq. (10). Hence the quantum Fisher information becomes

$$\mathcal{F}_Q(\tau,\mathcal{E}) = \frac{2\gamma t_{\rm tot}/(2n_T+1)}{\gamma\tau[1-(1-1/\mathcal{E})e^{-\gamma\tau}]}|D(\tau)|^2. \tag{E2}$$

The equation for the optimal time that maximizes this expression is

$$\frac{e^{\gamma\tau_{\rm opt}} - 1 + \gamma\tau_{\rm opt} - 2\gamma\tau_{\rm opt}(e^{\gamma\tau_{\rm opt}}-1)\left[\frac{d\ln|D(\tau_{\rm opt})|}{d(\gamma\tau_{\rm opt})}\right]}{-1+\gamma\tau_{\rm opt} + 2\gamma\tau_{\rm opt}\left[\frac{d\ln|D(\tau_{\rm opt})|}{d(\gamma\tau_{\rm opt})}\right]} = \frac{1}{\mathcal{E}}. \tag{E3}$$

We obtain now from this equation the optimal measurement times for constant forces and for resonant forces under the rotating-wave approximation.

1. Constant force

For a constant force, one gets from Eq. (28),

$$|D[t,t+\tau]|^2 = \frac{2\omega^2 e^{-\gamma\tau/2}}{\gamma^2/4+\omega^2}\left[\cosh(\gamma\tau/2)-\cos(\omega\tau)\right], \quad \text{(E4)}$$

which indeed depends only on τ.

Expanding $d\ln|D(t,t+\tau)|/d(\gamma\tau)$ in powers of $\gamma\tau$, one gets

$$\frac{d\ln|D(t,t+\tau)|}{d(\gamma\tau)} = \frac{1}{\gamma\tau} - \frac{1}{4} + \frac{\gamma\tau}{48}\left[1-4\left(\frac{\omega}{\gamma}\right)^2\right] + O[(\gamma\tau)^3]. \quad \text{(E5)}$$

Substituting this expansion into Eq. (E3), one gets

$$\frac{1}{\mathcal{E}} = \frac{1}{24}\left(1+\frac{4\omega^2}{\gamma^2}\right)(\gamma\tau_{\rm opt})^3 + O[(\gamma\tau_{\rm opt})^5], \quad \text{(E6)}$$

leading to Eq. (44). Note that for general conditions, $\tau_{\rm opt}$ and $\delta f_{\mathcal{E}}$ can be found by solving numerically Eq. (E3).

2. Rotating-wave approximation

In the RWA, $[4(\omega/\gamma)\tanh(\gamma\tau/4) \gg 1]$, the function $|D(t,t+\tau)|$ is well approximated by Eq. (50), so that $d\ln|D_{\rm RWA}(\tau)|/d(\gamma\tau)$ expands to

$$\frac{d\ln|D_{\rm RWA}(\tau)|}{d(\gamma\tau)} = \frac{1}{\gamma\tau} - \frac{1}{4} + \frac{\gamma\tau}{48} + O[(\gamma\tau)^3]. \quad \text{(E7)}$$

Substituting this into Eq. (E3) we find

$$\frac{1}{24}(\gamma\tau_{\rm opt})^3 + O[(\gamma\tau_{\rm opt})^5] = \frac{1}{\mathcal{E}}, \quad \text{(E8)}$$

leading to the optimal time

$$\gamma\tau_{\rm opt} \simeq \left(\frac{24}{\mathcal{E}}\right)^{1/3}. \quad \text{(E9)}$$

Hence Eq. (E2) becomes

$$\mathcal{F}_Q(\tau_{\rm opt},\mathcal{E}) = \frac{\omega^2(t_f-t_0)}{2\gamma(2n_T+1)}\left[1-\frac{1}{4}\left(\frac{3}{\mathcal{E}}\right)^{2/3} + O(\mathcal{E}^{-1})\right], \quad \text{(E10)}$$

while the uncertainty $\delta f_{\mathcal{E}}$ is given by

$$\delta f_{\mathcal{E}} = \sqrt{\frac{m\hbar\omega^3}{\mathcal{F}_Q(\tau_{\rm opt},\mathcal{E})}} = \sqrt{2}\,\delta f_{\rm min}^{(NB)}\left[1+\frac{1}{8}\left(\frac{3}{\mathcal{E}}\right)^{2/3} + O(\mathcal{E}^{-1})\right]. \quad \text{(E11)}$$

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