Quantum Speed Limit for Physical Processes

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The evaluation of the minimal evolution time between two distinguishable states of a system is important for assessing the maximal speed of quantum computers and communication channels. Lower bounds for this minimal time have been proposed for unitary dynamics. Here we show that it is possible to extend this concept to nonunitary processes, using an attainable lower bound that is connected to the quantum Fisher information for time estimation. This result is used to delimit the minimal evolution time for typical noisy channels.

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Introduction.—Quantum mechanics imposes fundamental limits to the processing speed of any device as well as to the communication speed through any channel. Derivation of these basic limits usually assumes that such devices are noiseless, undergoing unitary evolutions [1–38]. The relevant (and often unwanted) influence of the environment on processing or information-transferring systems is thus frequently ignored. On the other hand, this influence, and in particular the decoherence speed, plays an essential role in fundamental physics, especially in the understanding of the quantum-to-classical transition [39]. Here we unify the description of both computation or communication speed and decoherence speed in a single framework, which deals with the maximal speed of evolution of quantum systems.

Although much work has been done on the subject since the first major result by Mandelstam and Tamm [1], scarce contributions [40–44] undertake nonunitary evolutions. In this Letter, we develop a method that allows one to derive useful saturable lower bounds for the minimal evolution time for general physical processes. Besides recovering, in the proper limits, previous findings such as the Mandelstam-Tamm bound [1], our result allows the study of experimentally more realistic open systems and the development of a systematic approach for tackling such nonunitary evolutions. This approach relies on variational techniques, allowing one to obtain nontrivial analytical approximations to the bounds in situations where exact calculations are too involved. We exemplify the usefulness of this bound by considering typical nonunitary quantum channels.

General bound for the minimal evolution time.—We present here a general lower bound on the time τ necessary for a quantum system, evolving under the action of some physical process, to reach a final state that has a distance $\mathcal{D}$ from its initial state.

Let $D[F_B(\hat{\rho}_1, \hat{\rho}_2)]$ be a metric on the space of quantum states that depends on $\hat{\rho}_1$, $\hat{\rho}_2$ solely via the Bures fidelity F_B,

$$F_B(\hat{\rho}_1, \hat{\rho}_2) := \left[\mathrm{tr}\left(\sqrt{\sqrt{\hat{\rho}_1}\hat{\rho}_2\sqrt{\hat{\rho}_1}}\right)\right]^2. \tag{1}$$

Consider now a smooth dynamical process in this space, parametrized by t, and leading to an evolution described by the density operator $\hat{\rho}(t)$, such that $D\{F_B[\hat{\rho}(t_1), \hat{\rho}(t_2)]\}$ [written as $D(t_1, t_2)$ in a shorthand notation] is a piecewise smooth function of t_1, t_2. A bound on $D(0, \tau)$ can be obtained in terms of the integral of the quantum Fisher information for time estimation $\mathcal{F}_Q(t)$ along the path determined by system evolution. $\mathcal{F}_Q(t)$ may be defined by $\mathcal{F}_Q(t) = \mathrm{Tr}[\hat{\rho}(t)\hat{L}^2(t)]$ [45], where the Hermitian operator $\hat{L}(t)$ is known as the symmetric logarithmic derivative operator, implicitly defined by $d\hat{\rho}(t)/dt = [\hat{\rho}(t)\hat{L}(t) + \hat{L}(t)\hat{\rho}(t)]/2$. In order to derive the bound on $D(0, \tau)$, one applies to this metric the triangle inequality, considering a division of the interval $(0, \tau)$ into infinitesimal pieces, and one uses the relation between the Bures fidelity and the quantum Fisher information $\mathcal{F}_Q(t)$ [45],

$$F_B(t, t + dt) = 1 - (dt)^2\mathcal{F}_Q(t)/4 + \mathcal{O}(dt)^3. \tag{2}$$

This equation attaches a physical meaning to the quantum Fisher information: the square root of $\mathcal{F}_Q(t)$ is proportional to the instantaneous speed of separation between two neighboring states $\hat{\rho}(t)$ and $\hat{\rho}(t + dt)$, and can be used to delimit the minimal statistical uncertainty in the estimation of the duration of a given physical process, as shown in Ref. [45].

One gets then a general implicit lower bound on the evolution time τ, valid for arbitrary physical processes and any metric dependent on the Bures fidelity (see Supplemental Material [46]):

$$\sqrt{\frac{d^2D(F_B)/dF_B^2}{2[dD(F_B)/dF_B]^3}}\Bigg|_{F_B\to 1} D(0, \tau) \leq \int_0^\tau \sqrt{\frac{\mathcal{F}_Q(t)}{4}}dt, \tag{3}$$

where the notation $D(F_B)$ makes explicit the dependence of the metric on F_B. The argument of the square root on the left-hand side of the above inequality is proportional to the curvature of $D(F_B)$ at $F_B = 1$ (see Supplemental Material [46]). Notice that this bound is invariant by a rescaling $D' = kD$.

One should note that the right-hand side of Eq. (3) is the Bures length, as defined by Uhlmann [47], of the actual path followed by the state of the system $\hat{\rho}(t)$. On the other hand, it has also been shown in Ref. [47] that the Bures length of a geodesic joining two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ is $\arccos\sqrt{F_B(\hat{\rho}_1, \hat{\rho}_2)}$ [48], which defines a natural distance $\mathcal{D}$ between the two states. Inserting this expression into the left-hand side of (3), one obtains

$$\mathcal{D} := \arccos\sqrt{F_B[\hat{\rho}(0), \hat{\rho}(\tau)]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}dt. \quad (4)$$

Since it is always possible to find a dynamical process that joins $\hat{\rho}(0)$ and $\hat{\rho}(\tau)$ along a geodesic and saturates the above inequality, one finds that the left-hand side of (3), which depends only on the initial and final states, is maximized by $D(F_B) = \arccos\sqrt{F_B}$. Therefore, this is the optimal choice for $D(F_B)$ in (3), and leads to an attainable bound for the minimum evolution time, valid for unitary or nonunitary processes. The above discussion makes it clear that this bound is attained if and only if the evolution occurs on a geodesic, which is, incidentally, the same condition for attainability of the Mandelstam-Tamm bound for unitary processes with time-independent Hamiltonians [14]. This is an important requirement of quantum speed limits.

Although Eq. (2), proposed in Ref. [45], corresponds to a differential form of the right-hand side of Eq. (4), the integration of this infinitesimal distance is far from trivial. For instance, the Bures distance $D_{\text{Bures}}(0, \tau) = \sqrt{2}\sqrt{1 - \sqrt{F_B(\hat{\rho}(0), \hat{\rho}(\tau))}}$, mentioned in Ref. [45], coincides with the natural distance $\mathcal{D}$ for $\tau \to 0$, but does not lead to an achievable upper bound for finite values of τ.

Let us first consider a unitary evolution dictated by an operator $\hat{U}(t)$, which leads to a simple analytical expression for the quantum Fisher information. In this case, $\mathcal{F}_Q(t) = 4\langle\Delta\hat{H}^2(t)\rangle/\hbar^2$ [49], where $\langle\Delta\hat{H}^2(t)\rangle$ is the variance in the initial state of a Hermitian operator $\hat{H}(t)$ defined as

$$\hat{H}(t) := \frac{\hbar}{i}\frac{d\hat{U}^\dagger(t)}{dt}\hat{U}(t). \quad (5)$$

For a time-independent Hamiltonian $\hat{H}$, with $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$, then $\hat{H}(t) = \hat{H}$, and for $\mathcal{D} = \pi/2$ (orthogonal states), inequality (4) leads to the Mandelstam-Tamm bound: $\tau \geq (\pi\hbar/2)/\sqrt{\langle\Delta\hat{H}^2\rangle}$. Equation (4) also recovers the known implicit bounds on τ for time-dependent Hamiltonians [9,36].

For nonunitary evolutions, bound (4) can be hard to evaluate analytically since the quantum Fisher information may be difficult to calculate. In these situations, it is convenient to resort to the following purification procedure [50,51], which allows one to rely on the simple form of $\mathcal{F}_Q(t)$ for unitary processes.

To each system of interest S, represented by the density operator $\hat{\rho}_S$, one assigns an environment E, such that the dynamics of $\hat{\rho}_S$ results from a unitary evolution, corresponding to an operator $\hat{U}_{S,E}(t)$, of a pure state of the enlarged system $S + E$. The quantum Fisher information of $S + E$ is an upper bound to the quantum Fisher information of system S, since, from the point of view of parameter-estimation theory, $S + E$ does not contain less information about the parameter t than S alone. There are, in fact, infinitely many different evolutions of $S + E$ corresponding to the same evolution of system S, each of those leading to a possibly different value of the quantum Fisher information $\mathcal{C}_Q(t)$ of $S + E$. This freedom is integrally expressed by writing the purified unitary evolution in $S + E$ as $\hat{u}_E(t)\hat{U}_{S,E}(t)$, where $\hat{u}_E(t)$ is any unitary operator acting only on E. Defining $\hat{\mathcal{H}}_{S,E}(t)$ by inserting the evolution operator $\hat{u}_E(t)\hat{U}_{S,E}(t)$ into (5), one can write $\mathcal{C}_Q(t) = 4\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle/\hbar^2$. Then, for any upper bound $\mathcal{C}_Q(t)$ to $\mathcal{F}_Q(t)$, one can obtain an implicit lower bound to the evolution time τ, given by

$$\mathcal{D} \leq \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}dt = \int_0^\tau \sqrt{\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle}/\hbar dt. \quad (6)$$

Since $\mathcal{C}_Q(t)$ can be straightforwardly evaluated, the above bound may be easier to handle than bound (4). However, it can only be tight when $\mathcal{C}_Q(t) = \mathcal{F}_Q(t)$, in which case it reduces to bound (4). In fact, as it was shown in Refs. [50,51], it is always possible to fulfill this condition by minimizing $\mathcal{C}_Q(t)$ over all operators $\hat{u}_E(t)$, for given $\hat{U}_{S,E}(t)$. As $\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle$ only depends on $\hat{u}_E(t)$ through $\hat{h}_E(t)$,

$$\hat{h}_E(t) := \frac{\hbar}{i}\frac{d\hat{u}^\dagger_E(t)}{dt}\hat{u}_E(t), \quad (7)$$

the minimization can be performed with respect to $\hat{h}_E(t)$ [50]. Notice that in some practical situations, it can be advantageous to restrict the set of operators $\hat{h}_E(t)$ over which the optimization is done in order to obtain more tractable, albeit still useful bounds on τ. In the following, we present examples that illustrate the power and usefulness of this approach.

Amplitude-damping channel.—Let S be a two-state system (states $\{|0\rangle, |1\rangle\}$), and E its environment, which is chosen to start in state $|0\rangle_E$. The amplitude-damping channel is described by the map

$$|0\rangle|0\rangle_E \to |0\rangle|0\rangle_E, \quad (8a)$$

$$|1\rangle|0\rangle_E \to \sqrt{P(t)}|1\rangle|0\rangle_E + \sqrt{1 - P(t)}|0\rangle|1\rangle_E, \quad (8b)$$

where the state $|1\rangle_E$ is orthogonal to $|0\rangle_E$, and the time dependence of $P(t)$ reflects the damping dynamics. We consider here the paradigmatic exponential decay, with rate γ, so that $P(t) = e^{-\gamma t}$. We note that, for the above map, the environment may also be considered as a qubit. This channel, which corresponds to a *nonunitary* evolution of S, can be described by the unitary evolution operator acting on $S + E$,

$$\hat{U}_{S,E}(t) = \exp[-i\Theta(t)(\hat{\sigma}_+\hat{\sigma}_-^{(E)} + \hat{\sigma}_-\hat{\sigma}_+^{(E)})], \qquad (9)$$

where $\hat{\sigma}_\pm$ and $\hat{\sigma}_\pm^{(E)}$ are raising and lowering operators acting, respectively, on the system and environment qubits, and $\Theta(t) = \arccos\sqrt{P(t)}$.

Setting $\hat{u}_E(t)$ as an identity operator and inserting the variance of $\hat{\mathcal{H}}_{S,E}(t)$, obtained from the unitary operator (9) via definition (5), into (6), it is straightforward to show that τ is bounded by

$$\gamma\tau \geq 2\,\mathrm{lnsec}(\mathcal{D}/\sqrt{\langle\hat{\sigma}_+\hat{\sigma}_-\rangle}). \qquad (10)$$

Notice that, for the above process, the distance of the evolved state from the initial state can reach at most $\sqrt{\langle\hat{\sigma}_+\hat{\sigma}_-\rangle}\pi/2$. Bound (10) saturates either for S initially in the ground state ($\langle\hat{\sigma}_+\hat{\sigma}_-\rangle = 0$), when the system does not evolve at all, or for S initially in the excited state ($\langle\hat{\sigma}_+\hat{\sigma}_-\rangle = 1$), meaning that we have already chosen the purified evolution that yields $\mathcal{C}_Q(t) = \mathcal{F}_Q(t)$ for these situations. Furthermore, the fact that the bound is saturated implies that the amplitude-damping channel connects the states $|1\rangle$ and $|0\rangle$ along a geodesic path, which includes mixed states. This remains valid for any monotonically decreasing $P(t)$, with $P(0) = 1$, since, under this condition, changing the form of $\Theta(t)$ in (9) corresponds to a mere rescaling of the time parameter, which does not change the path in state space followed by a given initial state.

Markovian dephasing.—System S is again a single qubit whose nonunitary evolution is described by a map that makes use, as before, of an ancilla qubit starting in state $|0\rangle_E$,

$$|0\rangle|0\rangle_E \to e^{-i\omega_0 t/2}[\sqrt{P(t)}|0\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E],$$
$$|1\rangle|0\rangle_E \to e^{i\omega_0 t/2}[\sqrt{P(t)}|1\rangle|0\rangle_E - \sqrt{1-P(t)}|1\rangle|1\rangle_E], \qquad (11)$$

where $\hbar\omega_0$ is the energy difference between the qubit levels, $P(t) := (1 + e^{-\gamma t})/2$, and γ is the phase-decay constant. The corresponding evolution operator is

$$\hat{U}_{S,E}(t) = e^{-i\omega_0 t\hat{Z}/2}e^{-i\arccos\sqrt{P(\gamma t)}\hat{Z}\hat{Y}^{(E)}}, \qquad (12)$$

where $\hat{Z}$ and $\hat{Y}^{(E)}$ are Pauli operators acting on the system and on the environment qubits, respectively.

In order to find the best possible bound on τ within our approach, we now minimize $\mathcal{C}_Q(t)$ over the whole set of Hermitian 2×2 operators $\hat{h}_E(t)$. The minimum, written in terms of the (constant) variance of $\hat{Z}$, is [46]

$$\mathcal{C}_Q^{\mathrm{opt}}(t) = \langle\Delta\hat{Z}^2\rangle[\omega_0^2 e^{-2\gamma t} + \gamma^2(e^{2\gamma t} - 1)^{-1}], \qquad (13)$$

which reduces to $\mathcal{F}_Q(t)$ for pure initial states, in which case bound (6) reduces to (4). The above equation leads to an implicit bound on τ, given, in terms of elliptic integrals of the second kind $E(y, k)$, by

$$\mathcal{D} \leq \frac{1}{2}\sqrt{\langle\Delta\hat{Z}^2\rangle}\sqrt{r^2+1}\Big[E\Big(\frac{\pi}{2}, \frac{r}{\sqrt{r^2+1}}\Big) - E\Big(\arcsin e^{-\gamma\tau}, \frac{r}{\sqrt{r^2+1}}\Big)\Big], \qquad (14)$$

with $r := \omega_0/\gamma$. Equation (14) consistently guarantees the eigenstates of $\hat{Z}$ not to evolve. This bound is compared to an exact calculation of $\mathcal{D}$ in Fig. 1, which shows that it stays close to the exact result up to the first minimum of the latter. Another feature of (14) is that it captures the fact that the evolved and initial states never become orthogonal for r under a critical value $r_{\mathrm{crit}} \simeq 2.6$ [46].

In the extreme cases $\gamma \to 0$ and $\omega_0 \to 0$, inequality (14) yields simple analytical expressions for the bound on the minimal time. For the former, the Mandelstam-Tamm bound [2,6] is recovered, since the process becomes unitary, and for the latter

$$\gamma\tau \geq \mathrm{lnsec}\Big(2\mathcal{D}/\sqrt{\langle\Delta\hat{Z}^2\rangle}\Big), \qquad (15)$$

which saturates for pure initial states with $\langle\Delta\hat{Z}^2\rangle = 1$. In this situation, pure Markovian dephasing ($\omega_0 \to 0$) links a pure state to a fully mixed state through a geodesic path in state space. Notice that, for the above process, the distance of the evolved state from the initial state can reach at most $\sqrt{\langle\Delta\hat{Z}^2\rangle}\pi/4$.

Minimum evolution time and entanglement.—We now investigate the effect of subsystem correlations on the evolution speed of a compound system. We consider the Markovian dephasing of an N-qubit system where each

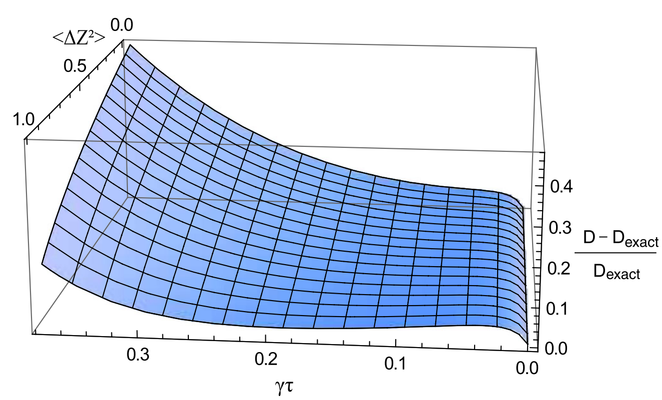

FIG. 1 (color online). Relative difference between bound on $\mathcal{D}$ for single-qubit dephasing (14) and the exact result, as a function of the dimensionless time $\gamma\tau$, for different values of $\langle\Delta\hat{Z}^2\rangle$ (different initial states), with $r = 8$.

qubit interacts only with its own environment, as described by (11), and compare how different initial-state correlations (possibly entanglement) affect the evolution speed. The evolution operator is $\hat{u}_E(t)\hat{U}_{S,E}(t)$, with

$$\hat{U}_{S,E}(t) = \prod_{i=1}^{N} e^{-i\omega_0 t\hat{Z}_i/2} e^{-i\arccos\sqrt{P(\gamma t)}\hat{Z}_i\hat{Y}_i^{(E)}}, \quad (16)$$

where $\hat{Z}_i$ ($\hat{Y}_i^{(E)}$) is a Pauli operator acting on the ith system (environment) qubit. Since $\hat{h}_E(t)$ now belongs to a $2^N \times 2^N$ space, the full minimization of $\mathcal{C}_Q(t)$ is rather cumbersome for large values of N. Hinging on the symmetry of the system, we resort instead to minimization over a three-parameter family of Hermitian operators:

$$\hat{h}_E(t) = \sum_{i=1}^{N}[\alpha(t)\hat{X}_i^{(E)} + \beta(t)\hat{Y}_i^{(E)} + \delta(t)\hat{Z}_i^{(E)}], \quad (17)$$

where $\alpha(t)$, $\beta(t)$, and $\delta(t)$ are optimization variables. We get then [46]

$$\mathcal{C}_Q^{\text{opt}}(t) = \langle\Delta\hat{\mathcal{Z}}^2\rangle\left[\frac{\omega_0^2 N^2}{Nq(e^{2\gamma t}-1)+1} + \frac{\gamma^2 N/q}{e^{2\gamma t}-1}\right], \quad (18)$$

where $q := \langle\Delta\hat{\mathcal{Z}}^2\rangle/(1-\langle\hat{\mathcal{Z}}\rangle^2)$, $\hat{\mathcal{Z}} = \sum_j \hat{Z}_j/N$, and the averages are taken in the initial state. We note that $0 \le q \le 1$; for a separable state, $q \le 1/N$ (equality if symmetrical on the N qubits). The values $q = 1$ and $\langle\Delta\hat{\mathcal{Z}}^2\rangle = 1$, achievable for the entangled state $[|0\ldots0\rangle + e^{i\phi}|1\ldots1\rangle]/\sqrt{2}$, yield a lower bound valid for any initial state. For $q = 1$, the bound on the minimum evolution time scales as $\tau \sim 1/N$ throughout, a prediction validated by exact calculations with the above entangled state; see Supplemental Material [46].

For separable states, on the other hand, the lower bound goes from a $\tau \sim 1/\sqrt{N}$ dependence for $\gamma\sqrt{N} \ll \omega_0$ to $\sim 1/N$ for $\gamma\sqrt{N} \gg \omega_0$, as shown in Fig. 2. This transition to faster behavior can be corroborated via direct calculations on symmetric, separable states [46].

This is a striking result, clearly distinct from the one corresponding to unitary evolution. It has already been seen in the literature [22–26] that, for unitary processes, entanglement is a resource that enhances the speed of evolution, so that the separation time improves from a $\tau \sim 1/\sqrt{N}$ scaling (separable, slow state) to $\tau \sim 1/N$ (entangled, fast state). However, for the nonunitary evolution considered here, the minimum evolution time for separable states, while scaling with $1/\sqrt{N}$ for small N, eventually scales as $1/N$ for $\gamma\sqrt{N} \gg \omega_0$, no matter how small the dephasing rate. Under this condition, the evolution speeds of separable and entangled states scale in the same way with respect to the number of qubits.

Conclusion.—We have derived an attainable lower bound for the minimal evolution time of dynamical systems through a geometrical approach. This bound applies

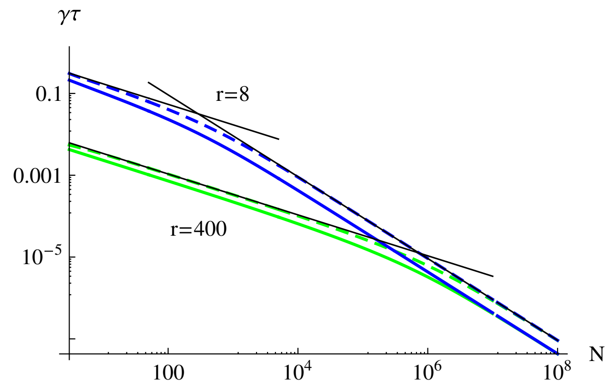

FIG. 2 (color online). Lower bound (solid curves) on time for separable, symmetric state with $\langle\Delta\hat{\mathcal{Z}}^2\rangle = 1$ to reach $\mathcal{D} \simeq 94\%$ of the maximal distance ($F_B = 1\%$), measured in dimensionless units $\gamma\tau$ as a function of number of qubits N, calculated numerically from (18). Results from exact calculations [46] are plotted for comparison (dashed curves). For the black (blue) curves, $r = 8$, for the light gray (green) curves, $r = 400$; the asymptotes are proportional to $1/N$ and $1/\sqrt{N}$.

to both unitary and nonunitary processes, and is obtained by comparing the actual path followed by the system in state space with the distance between the initial and final states along a geodesic path, defined by a metric that is expressed in terms of the Bures fidelity. Whenever the evolution between two states is along this geodesic, the bound is tight. Furthermore, it encompasses several special cases discussed in the literature, including unitary evolutions and mixed initial states.

This bound, expressed by Eq. (4), yields the proper speed limit for general physical processes and reduces, for unitary processes with time-independent Hamiltonians, to the Mandelstam-Tamm bound. This result invalidates claims that there is no general bound valid for all possible (unitary and nonunitary) quantum evolutions [40]. It is important to note that our general bound depends on the quantum Fisher information of the system, rather than the initial variance of the Hamiltonian of the system alone.

For situations when the general bound is too hard to evaluate, we have introduced a more tractable bound, based on a purification procedure that leads to attainable upper bounds for the quantum Fisher information.

The usefulness of this bound is exemplified by considering typical nonunitary quantum channels. For the amplitude channel, it leads to a tight bound, which evidences that the evolution between the initial and final orthogonal pure states is along a geodesic path though mixed states. For a dephasing channel, it yields very good lower bounds for the minimal evolution time between two nonorthogonal states. For N-qubit dephasing, the evolution speed-up due to entanglement of its subsystems, previously demonstrated for unitary evolution, is shown to hold, in the nonunitary case, also for separable states.

Our general result allows the estimation of the impact of the environment on the speed of quantum computation and information processing. It is also relevant for the estimation of thermalization and decoherence times.

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Note added.—After we submitted this paper, we noted the subsequent work [52], also proposing bounds for non-unitary processes, which lead to simple calculations in some situations, but are not saturable and do not recover the results for unitary evolutions.

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