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Stochastic Schrödinger equations in cavity QED: physical interpretation and localization

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Abstract. We propose physical interpretations, also valid for temperatures different from zero, for stochastic methods which have been developed recently to describe the evolution of a quantum system interacting with a reservoir. As opposed to the usual reduced density operator approach, which refers to ensemble averages, these methods deal with the dynamics of single realizations, and involve the solution of stochastic Schrödinger equations. These procedures have been shown to be completely equivalent to the master equation approach when ensemble averages are taken over many realizations. We show that these techniques are not only convenient mathematical tools for dissipative systems, but may actually correspond to concrete physical processes, for any temperature of the reservoir. We consider a mode of the electromagnetic field in a cavity interacting with a beam of two- or three-level atoms, the field mode playing the role of a small system and the atomic beam standing for a reservoir at finite temperature, the interaction between them being given by the Jaynes–Cummings model. We show that the evolution of the field states, under continuous monitoring of the state of the atoms which leave the cavity, can be described in terms of either the Monte Carlo wavefunction (quantum jump) method or a stochastic Schrödinger equation, depending on the system configuration. We also show that the Monte Carlo wavefunction approach leads, for finite temperatures, to localization into jumping Fock states, while the diffusion equation method leads to localization into states with a diffusing average photon number, which for sufficiently small temperatures are close approximations to mildly squeezed states. We prove analytically that, in the quantum jump situation, the system evolves in the mean towards a Fock state, even if an infinite number of photon-number amplitudes is present in the initial state.

Keywords: Stochastic Schrödinger equation, cavity QED, Monte Carlo, quantum jumps, localization

1. Introduction

The dynamics of dissipative quantum systems is frequently described through a master equation for the reduced density matrix, obtained by tracing out the degrees of freedom of the reservoir and making the Born–Markov approximation [1]. As usual in quantum mechanics, the corresponding solutions refer to ensembles of identical systems. In recent years, the attainment of low temperatures and low-dissipation regimes, as well as the improvement of detection techniques, has allowed the investigation of the dynamics of continuously monitored single quantum systems. Remarkable examples of these are single ions [2] or Bose–Einstein condensates [3] in electromagnetic traps, probed by laser beams, and electromagnetic fields in high-Q cavities, probed by beams of highly excited atoms (Rydberg atoms) [4]. This new generation of experiments, combined with the difficulties usually encountered in solving the master equation, have stimulated the development of new techniques, which seek to describe the dynamics of quantum dissipative systems by stochastic evolutions of the corresponding state vectors [5–15].

Basically two approaches have been proposed, one which involves random finite discontinuities in the dynamics of the system (Monte Carlo wavefunction (MCWF) or 'quantum jumps' method) [5, 9] and another for which the stochastic evolution is generated by a diffusive term in the Schrödinger equation for the state vector [7, 10]. These

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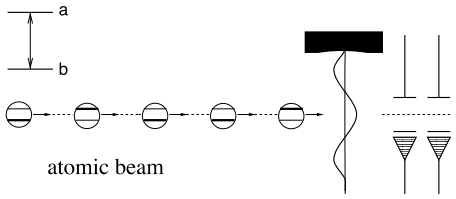

Figure 1. Physical realization of a quantum jump trajectory. A beam of two-level atoms crosses a resonant cavity.

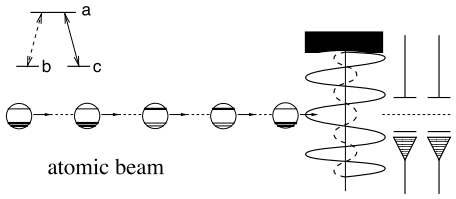

Figure 2. Physical realization of the homodyne stochastic Schödinger trajectory. A beam of three-level atoms crosses a resonant cavity, being subjected to an external classical field.

alternative techniques can be shown to be formally equivalent to the master equation approach [5, 9, 11], and sometimes do lead to a dynamical behaviour resembling the experimental monitoring of a single realization [9, 11, 12]. However, the possibility of attributing a physical interpretation to these techniques should not be overplayed: indeed, in some cases they are just mathematical tricks, with no relation to a concrete physical evolution of the system. The advantage of using them, from the numerical point of view, is that one deals with state vectors, instead of density matrices, thus reducing the total amount of matrix elements to be calculated. In addition, they may provide insights into the behaviour of dissipative systems. In fact, and because of these two points, they have been extensively applied to dissipative quantum systems, especially in the fields of quantum optics [8, 10–14, 16–20] and solid state physics [21]. In some cases, these methods have led to analytical descriptions of the dissipation process [16, 18, 22]. More recently, the MCWF method has been extended to non-Markovian interactions and to strong reservoir couplings beyond the Born and rotating-wave approximations [21], and also to nonlinear master equations [13, 14].

In this paper, we show that it is possible to interpret the above-mentioned stochastic evolutions in terms of continuous measurements made on concrete physical systems, for any temperature of the reservoir. Previous research along this direction includes the pioneering work of Carmichael [9], who established the connection between stochastic Schrödinger equations and photoelectric detection, and Wiseman and Milburn [11, 12], who developed physical interpretations for Schrödinger stochastic equations describing the evolution of a cavity mode in contact with a zero-temperature reservoir. These last two authors showed that the mode of the electromagnetic field is described by a quantum jump equation if the outgoing light is directly detected by a photodetector, while homodyne or heterodyne detection schemes lead to two different stochastic Schrödinger equations for the state of the field.

We also consider the time-dependent behaviour of a mode of the electromagnetic field in a cavity interacting with a reservoir (which may be associated to the continuum of modes of the field outside of the cavity, to which the internal field may be coupled via a semi-transparent mirror). Our interpretations rely on the fact that the same master equation is obtained for quite different reservoir models. We thus take as the reservoir an atomic beam which crosses the cavity, interacting resonantly with the field inside it. We show that the Monte Carlo or quantum jump method can be reproduced by taking as the reservoir a beam of continuously monitored two-level atoms, prepared initially in a mixture of the two resonant states, in such a way that the population ratio between the excited and the ground state equals the Boltzmann factor (see figure 1). On the other hand, a Schrödinger diffusion equation is obtained when the reservoir is assumed to be made of three-level atoms with a twofold-degenerated ground state, prepared in the same kind of statistical mixture as before, but now with the two ground state levels placed in a coherent superposition. While a resonant exchange of energy is allowed between the cavity mode, the excited state and one of the ground state levels, the transition between the other ground state level and the excited state is assumed to be driven by an external (essentially classical) field (see figure 2).

In order to further explore the physics underlying these approaches, we calculate the evolution of an initial field in the cavity under continuous monitoring, for different types of initial states of the field, with the two different methods mentioned above. We find that for temperatures different from zero the two approaches lead to two kinds of localization in state space. For the quantum jump method, the state of the system approaches a Fock state, which suffers quantum jumps in such a way that the average distribution in time of the number of photons satisfies the thermal distribution. On the other hand, the diffusion equation leads to states which, for sufficiently low temperatures (average number of thermal photons smaller than one), are quite close to mildly squeezed states. Even though localized in phase space, these states have a diffusive behaviour, eventually spanning a region of the phase space in such a way that again the time-averaged photon statistics coincides with the thermal distribution. We will show that our atomic model for the reservoir allows a simple interpretation of these localization phenomena, which extend to finite temperatures the discussion made by Garraway and Knight [16], and illustrates the general localization properties of quantum state diffusion equations demonstrated by Gisin and Percival [10].

The numerical simulations displayed in this paper involve, of course, initial states with a finite number of amplitudes in the Fock-state basis. This leaves open the question of whether the localization into Fock states would hold if one started with a state having an infinite number of components. We treat this problem analytically, and show that the system evolves in the mean towards a Fock state, for any initial state of the field.

In section 2, we review the stochastic approach to dissipative systems. In section 3 we propose a physical interpretation for the Monte Carlo quantum jump approach to the problem of field dissipation in cavity QED, for any reservoir temperature, while in section 4 we show how to

interpret physically a description of the same problem based on a stochastic Schrödinger equation. In section 5, we display our numerical results, and show that, depending on the physical procedure used to monitor continuously the field in the cavity, one may get localization in state space. We also prove analytically, in section 5, that the localization into Fock states occurs in the mean, for any initial state of the field. Our conclusions are summarized in section 6, while details of the calculations are displayed in the appendices.

2. Stochastic Schrödinger equations and dissipative systems

A wide class of master equations describing the evolution of dissipative quantum systems can be written in the Lindblad form [23]:

$$\dot{\rho}_S = \mathcal{L}\rho_S, \tag{1}$$

where

$$\mathcal{L} = \mathcal{L}_0 + \sum_n \mathcal{L}_n, \tag{2}$$

$$\mathcal{L}_0\rho_S = \frac{\mathrm{i}}{\hbar}[\rho_S, H_S], \tag{3}$$

$$\mathcal{L}_n\rho_S = -\tfrac{1}{2}[C_n^\dagger C_n\rho_S + \rho_S C_n^\dagger C_n] + C_n\rho_S C_n^\dagger, \tag{4}$$

ρ_S is the reduced density operator for the 'small' system S (obtained by tracing out the degrees of freedom of the reservoir R from the density operator for the full system $S + R$), and H_S describes the Hamiltonian evolution of the small system S in the interaction picture. The operators C_n act on the space of states of the small system S, and express the interaction of S with the reservoir R. The number of them depends on the nature of the problem.

An example of such an equation is the master equation for a field in a lossy cavity, at temperature T, given in the interaction picture by

$$\frac{\mathrm{d}\rho_f}{\mathrm{d}t} = \Gamma\overline{n}(a^\dagger\rho_f a - \tfrac{1}{2}aa^\dagger\rho_f - \tfrac{1}{2}\rho_f aa^\dagger) + \Gamma(1+\overline{n})(a\rho_f a^\dagger - \tfrac{1}{2}a^\dagger a\,\rho_f - \tfrac{1}{2}\rho_f a^\dagger a), \tag{5}$$

where a and $a^\dagger$ are the photon annihilation and creation operators, respectively, $\overline{n}$ is the average number of thermal photons, given by Planck's distribution, and $\Gamma = 1/t_{\mathrm{cav}}$, where t_{cav} is the damping time. In this case, one could set

$$C_1 \equiv \sqrt{\Gamma(1+\overline{n})}a, \qquad C_2 \equiv \sqrt{\Gamma\overline{n}}a^\dagger. \tag{6}$$

A formal solution of equation (1) is

$$\rho(t) = \exp(\mathcal{L}t)\rho(0). \tag{7}$$

Let us define

$$J_n\rho = C_n\rho C_n^\dagger, \tag{8}$$

and write

$$\rho(t) = \exp\left\{\mathcal{L}_0 t + \sum_n [J_n + (\mathcal{L}_n - J_n)]t\right\}\rho(0). \tag{9}$$

Note that

$$(\mathcal{L}_n - J_n)\rho_S = -\tfrac{1}{2}(C_n^\dagger C_n\rho_S + \rho_S C_n^\dagger C_n). \tag{10}$$

Applying Dyson's expansion to equation (9), we get:

$$\rho(t) = \sum_{m=0}^{\infty}\int_0^t \mathrm{d}t_m \int_0^{t_m}\mathrm{d}t_{m-1}\ldots\int_0^{t_2}\mathrm{d}t_1\left\{S(t-t_m)\times\left(\sum_n J_n\right)S(t_m-t_{m-1})\ldots\left(\sum_n J_n\right)S(t_1)\right\}\rho(0), \tag{11}$$

where

$$S(t) = \exp\left\{\left[\mathcal{L}_0 + \sum_n(\mathcal{L}_n - J_n)\right]t\right\}. \tag{12}$$

Equation (11) may be rewritten in the following way:

$$\rho(t) = \sum_{m=0}^{\infty}\sum_{\{n_i\}}\int_0^t \mathrm{d}t_m\int_0^{t_m}\mathrm{d}t_{m-1}\ldots\int_0^{t_2}\mathrm{d}t_1 \times\{S(t-t_m)J_{n_m}S(t_m-t_{m-1})\ldots J_{n_1}S(t_1)\}\rho(0). \tag{13}$$

Each term in the above double sum can be considered as a quantum trajectory, the reduced density operator at time t being given by the sum over all possible quantum trajectories [9]. For each of these trajectories, equation (13) shows that the evolution of the system can be considered as a succession of quantum jumps, associated to the operators J_n, interspersed by smooth time evolutions, associated with the operators $S(t)$. The probability of each trajectory is given by the trace of the corresponding term in equation (13).

From equations (10) and (12), we can write:

$$S(t)\rho = N(t)\rho N(t)^\dagger, \tag{14}$$

where

$$N(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}H_S t - \frac{t}{2}\sum_n(C_n^\dagger C_n)\right]. \tag{15}$$

Therefore, if ρ is a pure state, then $S(t)\rho$ is also a pure state. The same is true for $J_n\rho$, with J_n defined by equation (8). This implies that a pure state remains pure, when a single quantum trajectory is considered. Note also that the evolution between jumps is given by the non-unitary operator $N(t)$.

It is clear from equation (9) that different choices of the jump operators are possible. These different choices correspond to different decompositions in terms of quantum trajectories of the time evolution of the density operator ρ_S and, eventually, to different experimental schemes leading to the continuous monitoring of the evolution of the system. It is precisely due to this continuous monitoring that an initial pure state remains pure, since no information is lost in this situation: for a field in a cavity, this continuous monitoring amounts to accounting for every photon gained or lost by the field, due to its interaction with the reservoir.

We will discuss now two different realizations of the reservoir, for a field in a cavity, which will lead to a Monte Carlo quantum jump equation, for the first realization, and to a Schrödinger equation with stochastic terms, for the second one.

3. Simulation of a Monte Carlo SSE

We exhibit in this section a physical realization of the Monte Carlo method. The corresponding experimental scheme is shown in figure 1. A monokinetic atomic beam plays the

role of a reservoir R and crosses a lossless cavity, interacting with one mode of the electromagnetic field. The cavity mode plays the role of a small system S. The atoms, regularly spaced along the atomic beam, are prepared in one of two Rydberg states: an upper state $|a\rangle$ or a lower state $|b\rangle$. The transition frequency ω between these two states is assumed to be resonant with the cavity mode. A similar model of reservoir was adopted in section 16.1 of [24].

The state of the atoms is measured by a detector just at the exit of the cavity. The ratio between the flux of upper state atoms r_a and the lower state atoms r_b before their entrance into the cavity is chosen so that

$$\frac{r_a}{r_b} = \mathrm{e}^{-\hbar\omega/k_BT} \equiv \frac{\overline{n}}{1+\overline{n}}, \tag{16}$$

where $\hbar\omega$ is the difference in energy between $|a\rangle$ and $|b\rangle$, and, as will be shown in the next paragraphs, T is the reservoir temperature. The constant k_B represents the Boltzmann constant and $\overline{n}$, given by Planck's formula ($\overline{n} = [\exp(\hbar w/k_BT)-1]^{-1}$), is the mean occupation of the modes with energy $\hbar w$ in a bath at temperature T.

We now analyse the time evolution of the state vector $|\Psi(t)\rangle$ of S, under the continuous measurement of the atoms after they leave the cavity. We also assume that one knows the state of each atom before it interacts with the cavity. This may be achieved by selectively exciting the atoms to $|a\rangle$ or $|b\rangle$, according to the proportion given by equation (16). We will adopt the following simplifying assumptions: (a) the atom–field interaction time τ is the same for all atoms; (b) the spatial profile of the electric field is constant; (c) the cavity is perfect, i.e., the field state is changed only by the atoms; (d) the atom–field coupling constant λ and the interaction time τ are both small, so that the atomic state rotation is very small; (e) the rotating-wave and dipole approximations will be used; and (f), according to the statements (d) and (e), quantum cooperative effects will be neglected. In this case the interaction Hamiltonian in the interaction picture will be:

$$H = \hbar\lambda(|b\rangle\langle a|a^\dagger + |a\rangle\langle b|a). \tag{17}$$

The operators a and $a^\dagger$ are annihilation and creation operators, acting on the space of states of the field mode. Just before the ith atom enters the cavity, the state describing the combined system (atom i + field) is given by

$$|\Psi_{a-f}(t_i)\rangle = |\Psi(t_i)\rangle \otimes |\Psi_a(t_i)\rangle. \tag{18}$$

Here $|\Psi_a(t_i)\rangle = |a\rangle$ or $|\Psi_a(t_i)\rangle = |b\rangle$, depending on the state to which the atom was excited just prior to entering the cavity.

At time $t_i + \tau$, the atom–field state vector, up to second order in τ, is given by:

$$|\tilde{\Psi}_{a-f}(t_i+\tau)\rangle = \left(1 - \mathrm{i}\lambda\tau|b\rangle\langle a|a^\dagger - \mathrm{i}\lambda\tau|a\rangle\langle b|a - \frac{\lambda^2\tau^2}{2}|b\rangle\langle b|a^\dagger a - \frac{\lambda^2\tau^2}{2}|a\rangle\langle a|aa^\dagger\right)|\Psi_{a-f}(t_i)\rangle, \tag{19}$$

where the tilde indicates that the state vector is not normalized. The expansion (19) should be very good in view of condition (d). We assume that $(r_a + r_b)\tau < 1$, so that there is at most one atom inside the cavity at each instant of time. After this atom exits the cavity and is detected, one of the following four cases will be realized:

(i) The atom enters the cavity in state $|b\rangle$ and is detected in the same state. In this case, according to equation (19), the state of S at time $t = t_i + \tau$ will be given by

$$|\tilde{\Psi}(t_i+\tau)\rangle = \left(1 - \frac{\lambda^2\tau^2}{2}a^\dagger a\right)|\Psi(t_i)\rangle. \tag{20}$$

(ii) The atom enters the cavity in state $|a\rangle$ and it is detected in the same state $|a\rangle$. In this case,

$$|\tilde{\Psi}(t_i+\tau)\rangle = \left(1 - \frac{\lambda^2\tau^2}{2}aa^\dagger\right)|\Psi(t_i)\rangle. \tag{21}$$

(iii) The atom enters the cavity in the state $|b\rangle$ and it is detected in the state $|a\rangle$. In this case,

$$|\tilde{\Psi}(t_i+\tau)\rangle = -\mathrm{i}\lambda\tau a|\Psi(t_i)\rangle. \tag{22}$$

(iv) The atom enters the cavity in the state $|a\rangle$ and it is detected in the state $|b\rangle$. Then,

$$|\tilde{\Psi}(t_i+\tau)\rangle = -\mathrm{i}\lambda\tau a^\dagger|\Psi(t_i)\rangle. \tag{23}$$

Note that in the cases (i) and (ii) a small change in the state of 'S' takes place, while in the cases (iii) and (iv) a big change may happen (quantum jump). However, these last two cases are very rare, due to the small change of the atomic state during the interaction time.

We consider now the change of $|\Psi\rangle$ from t to $t+\delta t$, where the time interval δt is large enough so that many atoms go through the cavity during this time interval ($n_a = r_a\delta t \gg 1$, $n_b = r_b\delta t \gg 1$), and also much smaller than $t_{\mathrm{cav}}/\overline{n}\langle n\rangle$, where $\langle n\rangle$ is the average number of photons in the state. This last condition, as it will be seen later, implies that the probability of a quantum jump during δt is very small. In most of the time intervals δt the atoms will be detected at the same state they came in, since the transition probability is very small. The evolution of $|\Psi\rangle$ during these intervals will be given by:

$$|\tilde{\Psi}(t+\delta t)\rangle = \left(1 - \frac{\lambda^2\tau^2}{2}aa^\dagger\right)^{n_a}\left(1 - \frac{\lambda^2\tau^2}{2}a^\dagger a\right)^{n_b}|\Psi(t)\rangle = \left(1 - \frac{n_a\lambda^2\tau^2}{2}aa^\dagger - \frac{n_b\lambda^2\tau^2}{2}a^\dagger a\right)|\Psi(t)\rangle. \tag{24}$$

This result does not depend on the ordering of the upper-state and lower-state atoms. We also note that in the interaction picture the state vector does not evolve when there is no atom inside the cavity, since the only source of field dissipation is the interaction with the atomic beam.

Equation (24) displays the interesting property that the wavefunction of the system (and, consequently, the mean energy) may change even when there is no exchange of energy between the system and the measurement apparatus (represented by the atoms in the present case). An easy way to understand this effect physically is to imagine that all atoms are sent into the cavity in the lower state, and are detected in the same state after exiting the cavity, for a given realization of the system, which starts with a coherent state in the cavity. Then, even though there is no exchange of energy between the atoms and the field in the cavity, as time evolves the ground state component of the initial state should also increase, since the results of the measurements lead to an increasing probability that there is a vacuum state in the

cavity. In other words, the fact that there is no quantum jump, for that specific trajectory, provides us with information about the quantum state of the system, and this information leads to an evolution of the state. This is closely related to the quantum theory of continuous measurement [25,26] and also to quantum non-demolition measurement schemes proposed recently [27]. This problem is also very similar to that of a Heisenberg microscope in which even the unsuccessful events of light scattering produce a change in the quantum mechanical state of the particle [28].

We introduce now the following definitions:

$$\Gamma \equiv (r_b - r_a)\lambda^2\tau^2 = \frac{r_b}{1+\overline{n}}\lambda^2\tau^2 = \frac{r_a}{\overline{n}}\lambda^2\tau^2, \qquad (25)$$

$$C_1 \equiv \sqrt{\Gamma(1+\overline{n})}a, \qquad C_2 \equiv \sqrt{\Gamma\overline{n}}a^\dagger. \qquad (26)$$

Using these definitions and relation (16), equation (24) may be rewritten in the following way:

$$|\tilde{\Psi}(t+\delta t)\rangle = \left[1 - \frac{\delta t}{2}\sum_m C_m^\dagger C_m\right]|\Psi(t)\rangle. \qquad (27)$$

If an atom enters the cavity in state $|a\rangle$ and is detected in the state $|b\rangle$, the state vector of S suffers a 'quantum jump', and one photon is added to that system. On the other hand, a de-excitation in S occurs if an atom which entered in $|b\rangle$ is detected in the state $|a\rangle$. The probability of this event occuring may be calculated by using equations (26) and (22) or (23); thus, the probability of an excitation (action of $a^\dagger$) to occur between t and $t+\delta t$ is given by:

$$\delta p_1 = \delta t\langle\Psi(t)|C_1^\dagger C_1|\Psi(t)\rangle. \qquad (28)$$

The probability of a de-excitation (action of a) during this time interval is:

$$\delta p_2 = \delta t\langle\Psi(t)|C_2^\dagger C_2|\Psi(t)\rangle. \qquad (29)$$

The probabilities δp_1 and δp_2 are very low, so that the joint probability of having one excitation and one de-excitation during the same time interval δt is negligible. One may therefore write:

$$|\tilde{\Psi}(t+\delta t)\rangle = C_1^{\delta N_1}C_2^{\delta N_2}\left[1 - \frac{\delta t}{2}\sum_m C_m^\dagger C_m\right]|\Psi(t)\rangle. \qquad (30)$$

where δN_1 and δN_2 are equal to one or zero, with probabilities δp_1 and δp_2 for δN_1 and δN_2 to be equal to one, respectively. This may be represented by writing the statistical mean $M(\delta N_m) = \langle C_m^\dagger C_m\rangle\delta t$. Also, $\delta N_m\delta N_n = \delta N_m\delta_{nm}$. One should note that the instants of time in which the quantum jumps occur during the time interval δt are irrelevant, since the jump operators can be commuted through the no-jump evolution, the commutation producing an overall phase which goes away upon renormalization of the state. This can be easily seen by rewriting the no-jump evolution, during a time interval $\delta t_j < \delta t$, as an exponential:

$$1 - \frac{\delta t_j}{2}\sum_m C_m^\dagger C_m = \exp\left(-\frac{\delta t_j}{2}\sum_m C_m^\dagger C_m\right) + \mathrm{O}[(\delta t_j)^2], \qquad (31)$$

and using that

$$C_i\mathrm{e}^{-\frac{\delta t_j}{2}\sum_m C_m^\dagger C_m} = \mathrm{e}^{-\frac{\delta t_j}{2}\sum_m C_m^\dagger C_m}C_i\mathrm{e}^{\lambda_i}, \qquad (32)$$

where $\lambda_1 = -(\delta t_j/2)\Gamma(1+\overline{n})$ and $\lambda_2 = (\delta t_j/2)\Gamma\overline{n}$.

The results of the measurement may be simulated by picking random numbers. The state vector given by equation (30) may be normalized as follows:

$$|\psi(t+\delta t)\rangle = \left[\frac{C_1}{\sqrt{C_1^\dagger C_1}}\delta N_1 + \frac{C_2}{\sqrt{C_2^\dagger C_2}}\delta N_2 + (1-\delta N_1)(1-\delta N_2)\left(1 - \frac{\delta t}{2}\sum_m C_m^\dagger C_m\right) \times \left(1 - \delta t\sum_m\langle C_m^\dagger C_m\rangle\right)^{-\frac{1}{2}}\right]|\psi(t)\rangle. \qquad (33)$$

In the above equation, the first two terms represent the possible jumps, each normalized, as in the Monte Carlo method, and the last term is the no-jump evolution contribution, normalized with the corresponding prefactor that rules out the jumps. From equation (33) one gets for $|\mathrm{d}\psi(t)\rangle \equiv |\psi(t+\delta t)\rangle - |\psi(t)\rangle$:

$$|\mathrm{d}\psi(t)\rangle = \left\{\sum_m\left[\frac{C_m}{\sqrt{C_m^\dagger C_m}} - 1\right]\delta N_m - \frac{\delta t}{2}\sum_m(C_m^\dagger C_m - \langle C_m^\dagger C_m\rangle)\right\}|\psi(t)\rangle. \qquad (34)$$

4. Simulation of the homodyne SSDE

We show now that, by a suitable modification of the atomic configuration, it is also possible to interpret physically the stochastic Schrödinger equations in terms of continuous measurements made on atoms which cross the cavity containing the field. The corresponding scheme is shown in figure 2: a beam of three-level atoms with a degenerate lower state (states b and c) crosses the cavity, the field in the cavity being resonant with a transition between one of the two lower levels (say, level b) and the upper atomic state a, while a strong classical field connects the other lower state with the upper level (one may assume that both fields are circularly polarized, so that the cavity field cannot connect a and c, while the strong field does not induce transitions between a and b).

We also assume that the atom is prepared in either a coherent superposition of the two lower levels:

$$|\psi_{atom}\rangle = \frac{1}{\sqrt{2}}(|b\rangle + |c\rangle), \qquad (35)$$

or in the upper one, following a Boltzmann distribution corresponding to a temperature T for the atoms, which act as a reservoir for the quantum field in the cavity.

In the interaction picture, one can write:

$$H = \hbar g_{ac}(\varepsilon|a\rangle\langle c| + \varepsilon|c\rangle\langle a|) + \hbar g_{ab}(a^\dagger|b\rangle\langle a| + a|a\rangle\langle b|. \qquad (36)$$

We assume for simplicity that $g_{ac} = g_{ab} = g$, and that ε is real. The time evolution of the wavefunction, to second order

in the coupling constant is:

$$|\psi(t+\tau)\rangle = \left[1 - \frac{\mathrm{i}H\tau}{\hbar} - \frac{H^2\tau^2}{2\hbar^2}\right]|\psi(t)\rangle. \qquad (37)$$

As in the previous model, there are two possible quantum jump processes. The first one corresponds to the atom entering the cavity in the coherent superposition of lower states, and being detected in the upper state. After the measurement, the state of the field is given by:

$$|\psi(t+\tau)\rangle_f^{(b,c\to a)} = \frac{-\mathrm{i}g\tau}{\sqrt{2}}(\varepsilon + a)|\psi(t)\rangle_f. \qquad (38)$$

The corresponding probability of detecting an atom in $|a\rangle$, after a time interval δt, starting from the initial superposition state, is given by:

$$\delta p_1 = n_b \frac{g^2\tau^2}{2}\langle\psi_f(t)|(\varepsilon + a^\dagger)(\varepsilon + a)|\psi_f(t)\rangle, \qquad (39)$$

where $n_b \equiv r_b\delta t$, r_b being the rate of atoms injected in the superposition of the lower states.

The second jump process corresponds to the atom entering the cavity in the upper state $|a\rangle$, and being detected in the superposition of lower states. Then, the state of the field after the measurement is:

$$|\psi(t+\tau)\rangle_f^{(a\to b,c)} = \frac{-\mathrm{i}g\tau}{\sqrt{2}}(\varepsilon + a^\dagger)|\psi(t)\rangle_f. \qquad (40)$$

The corresponding probability is given by:

$$\delta p_2 = n_a \frac{g^2\tau^2}{2}\langle\psi_f(t)|(\varepsilon + a)(\varepsilon + a^\dagger)|\psi_f(t)\rangle, \qquad (41)$$

where $n_a = r_a\delta t$ is the number of atoms which enter the cavity in state $|a\rangle$, during the time interval δt.

This analysis suggests that the quantum jump operators corresponding to these two processes should be, respectively,

$$C_1 = \sqrt{\Gamma(1+\overline{n})}(\varepsilon + a) \qquad (42a)$$

and

$$C_2 = \sqrt{\Gamma\overline{n}}(\varepsilon + a^\dagger), \qquad (42b)$$

where

$$\Gamma \equiv (r_b - r_a)\frac{g^2\tau^2}{2} = \frac{r_b}{1+\overline{n}}\frac{g^2\tau^2}{2} = \frac{r_a}{\overline{n}}\frac{g^2\tau^2}{2}. \qquad (43)$$

Formally, these jump operators are retrieved by rewriting the master equation (5) in the following equivalent form:

$$\frac{\mathrm{d}\rho_f}{\mathrm{d}t} = (J_1 + J_2)\rho_f - \frac{\Gamma(1+\overline{n})}{2}[(a^\dagger a + 2\varepsilon a + \varepsilon^2)\rho_f + \rho_f(a^\dagger a + 2\varepsilon a^\dagger + \varepsilon^2)] - \frac{\Gamma\overline{n}}{2}[(aa^\dagger + 2\varepsilon a^\dagger + \varepsilon^2)\rho_f + \rho_f(aa^\dagger + 2\varepsilon a + \varepsilon^2)], \qquad (44)$$

with:

$$J_i = C_i\rho C_i^\dagger, \qquad i = 1, 2 \qquad (45)$$

being associated with the jumps, the operators C_i being now given by equations (42).

We derive now the stochastic Schrödinger equation that describes the present measurement scheme.

With the above jump operators, and using the expansion given by equation (13), we show in appendix A that the joint probability of getting m_1 and m_2 jumps corresponding respectively to the first and second processes described above is given by the following expression:

$$P_{m_1,m_2}(\Delta t) = \left[\exp\mu_1\frac{(\mu_1)^{m_1}}{m_1!}\right]\left[\exp\mu_2\frac{(\mu_2)^{m_2}}{m_2!}\right] \times \mathrm{Tr}\left\{\exp\beta'\left[1 + \frac{1}{\varepsilon}(m_1 a + m_2 a^\dagger)\right]\rho \times \left[1 + \frac{1}{\varepsilon}(m_1 a^\dagger + m_2 a)\right]\exp\beta^{\dagger\prime}\right\}, \qquad (46)$$

where:

$$\mu_1 = \Gamma\Delta t\varepsilon^2(1+\overline{n}),$$
$$\mu_2 = \Gamma\Delta t\varepsilon^2(\overline{n}), \qquad (47)$$
$$\beta' = -\frac{\Gamma\Delta t}{2}[a^\dagger a(2\overline{n}+1) + 2\varepsilon a(\overline{n}+1) + 2\varepsilon a^\dagger\overline{n} + \overline{n})].$$

From equations (46) and (47), one can readily find $\langle m_i\rangle$ and $\langle m_i^2\rangle$ for i = 1, 2.

Up to order $\varepsilon^{-3/2}$, one finds:

$$\langle m_i\rangle = \mu_i(1 + \tfrac{2}{3}\langle X_1\rangle), \qquad \langle m_i^2\rangle = \mu_i, \qquad (48)$$

with

$$X_1 \equiv \frac{a + a^\dagger}{2}. \qquad (49)$$

Going back to the definition of $S(t)$, one may write:

$$S(\Delta t) = N(\Delta t)\rho N^\dagger(\Delta t), \qquad (50)$$

in terms of a smooth evolution operator N that preserves pure states. This operator N is given by equation (15). with the jump operators C_m now given by equations (42). Now, if we consider a sequence of jumps (of the two kinds, in the present analysis) and evolutions, the state vector of the field will evolve according to:

$$|\tilde{\psi}\rangle_f(\Delta t) = N(\Delta t - t_m)C_2 N(t_m - t_{m-1})C_1 \ldots |\psi\rangle_f(0) = N(\Delta t)C_2^{m_2}C_1^{m_1}|\psi\rangle_f(0). \qquad (51)$$

In the last step, in deriving equation (51), we used that the commutators between the jump operators and the no-jump evolution produce overall phases, as in the Monte Carlo evolution given by equation (30).

Now, we consider m_i, $i = 1, 2$ as a couple of random variables with non-zero average, and write them as:

$$m_i = \langle m_i\rangle + \Delta W_i\frac{\sigma_i}{\sqrt{\Delta t}}, \qquad (52)$$

where the ΔW_i are two real and independent Wiener increments, with:

$$\langle\Delta W_i^2\rangle = \Delta t, \qquad i = 1, 2. \qquad (53)$$

From equations (51) and (52) and up to order $\varepsilon^{-3/2}$, we get the following homodyne stochastic Schrödinger

differential equation (HSSDE):

$$\Delta^{m_1,m_2}|\tilde{\psi}\rangle_f(\Delta t) = |\tilde{\psi}\rangle_f(\Delta t) - |\psi\rangle_f(0)$$
$$= \left\{\left[-\frac{\Gamma}{2}(1+\overline{n})a^\dagger a - \frac{\Gamma}{2}(\overline{n})aa^\dagger \right.\right.$$
$$\left. +2\Gamma\langle X_1\rangle(a(1+\overline{n}) + a^\dagger\overline{n})\right]$$
$$\left.\times\Delta t + a^\dagger\sqrt{\Gamma\overline{n}}\Delta W_2 + a\sqrt{\Gamma(1+\overline{n})}\Delta W_1\right\}|\psi\rangle_f(0). \quad (54)$$

At zero temperature, a typical quantum trajectory in this homodyne scheme is as follows:

(a) If one starts from a coherent state, the quantum jumps will only produce a multiplicative factor in the wave function of the field, a factor that can be absorbed in the normalization.
On the other hand, during the 'no-click' periods, the nature of the coherent state is preserved, changing only the coherent amplitude, all the way to the vacuum.
This situation has been previously studied [29] in the context of the continuous measurement theory of three-level atoms and two resonant fields, with the difference that in that work the number of detections was a predetermined quantity. However, the net result of the preservation of the coherent nature of the state of the field, along the trajectory, is the same.

(b) If we start with a Fock state, the quantum jumps will invariably produce a mixture of various Fock states, while the waiting or 'no-click' periods will only generate numerical factors in front of those Fock states.

In the finite-temperature case, the situation is more complex, since there will be also creation of photons, that will disturb an initial coherent state and produce further mixtures in the Fock state case.

A more detailed analysis of these various cases is described in the next section, devoted to the numerical simulation.

5. Numerical results and localization

We present now the numerical calculations corresponding to the two equations associated with the two measurement schemes discussed above. We consider in these calculations the general case in which the temperature of the reservoir is taken as different from zero.

5.1. Quantum jumps evolution

We consider first a simple example in which the initial state of the system is a Fock state with three photons. We assume that the temperature of the reservoir corresponds to an average number of photons also equal to three. The corresponding evolutions are exhibited in figure 3. The state of the system remains a Fock state, with a number of photons which keeps jumping between several values, in such a way that the average number of photons is equal to three. We have verified that the probability distribution for the number of photons is a Bose–Einstein distribution, as long as the observation is done over a sufficiently large time. Our model leads to an obvious physical interpretation for this behaviour: as each atom is detected the photon number jumps up or down, depending on the states of the incoming and outgoing atoms.

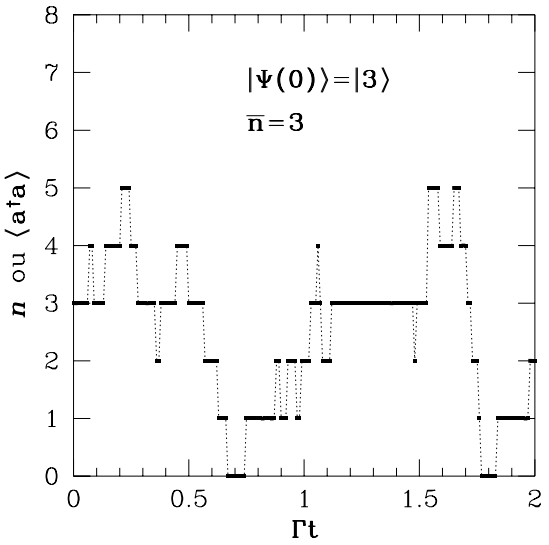

Figure 3. Quantum jumps for an initial Fock state with $n = 3$, the number of thermal photons being also equal to three. The set-up is the one shown in figure 1.

Figure 4 displays two different views of the evolution of the photon number population $|a_n|^2$ of an initial coherent state. These figures clearly exhibit the dual nature of the system dynamics, with quantum jumps interspersed by non-unitary evolutions. In the displayed realization, the vacuum component of the state increases until the first quantum jump occurs. This jump corresponds to the addition of a thermal photon to the system, leading to the disappearance of the vacuum component. The second jump corresponds to the absorption of a photon from the cavity field, leading to the reappearance of the vacuum state. The combination of the non-unitary evolution with the quantum jumps finally leads to a Fock state, which under the action of the reservoir keeps jumping, in such a way that the photon number distribution over a long time span reproduces the Bose–Einstein distribution. This process is illustrated in figure 5, which displays the time evolution of the Q distribution for the field, defined for each realization as $Q = |\langle\alpha|\psi\rangle|^2/\pi$, where $|\alpha\rangle$ is a coherent state with amplitude α. The initial Q distribution is a Gaussian, corresponding to the initial coherent state $|\alpha_0\rangle$, with $\alpha_0 = \sqrt{15/2}(1+\mathrm{i})$. This distribution evolves into the one corresponding to a Fock state, with a number of photons which keeps jumping around the thermal value $\overline{n} = 2$, in the same way as shown in figure 3. The convergence towards a Fock state admits a simple physical interpretation, if the state has a finite number of components in the Fock-state basis: if one has a sequence of absorbing interactions, so that the field goes to the vacuum state, from then on one would have the evolution considered in the previous paragraph. The same would happen if no change in the atomic state is detected after a long time, since in this case the field also evolves towards the vacuum state. More generally, one can see from figure 4 that the photon-number distribution is continually renormalized towards lower photon numbers, which makes it easier to reach the

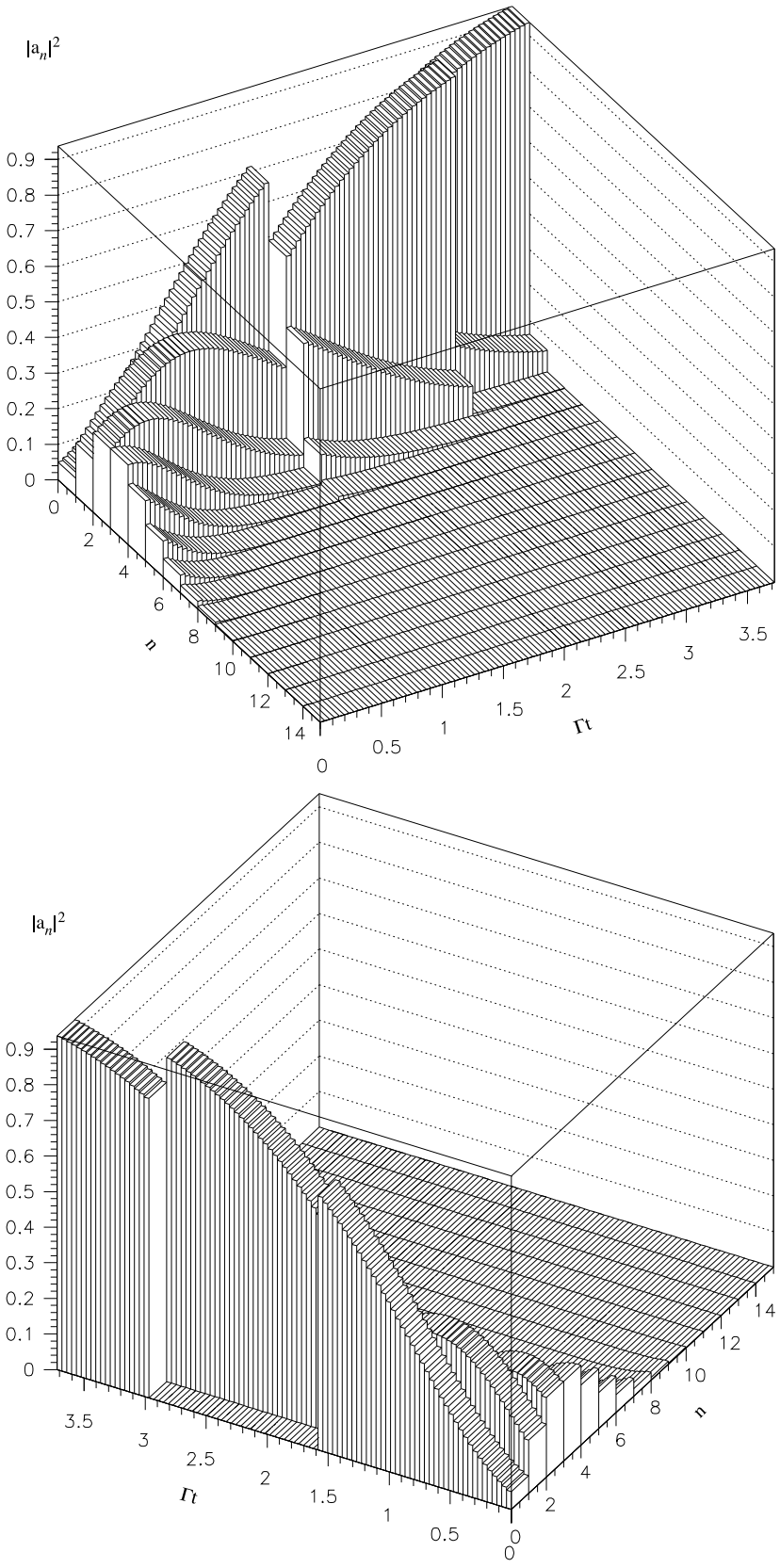

Figure 4. Two views of the evolution of an initial coherent state (average photon number equal to three), in the quantum jump approach. The temperature of the reservoir corresponds to a number of thermal photons equal to 0.2. At $\Gamma t = 1.52$ a photon is absorbed by the cavity mode, while around $\Gamma t = 3$ a photon is lost by the field in the cavity. Before the first jump, the amplitude of the coherent state decreases exponentially. After some jumps, the state becomes a jumping Fock state.

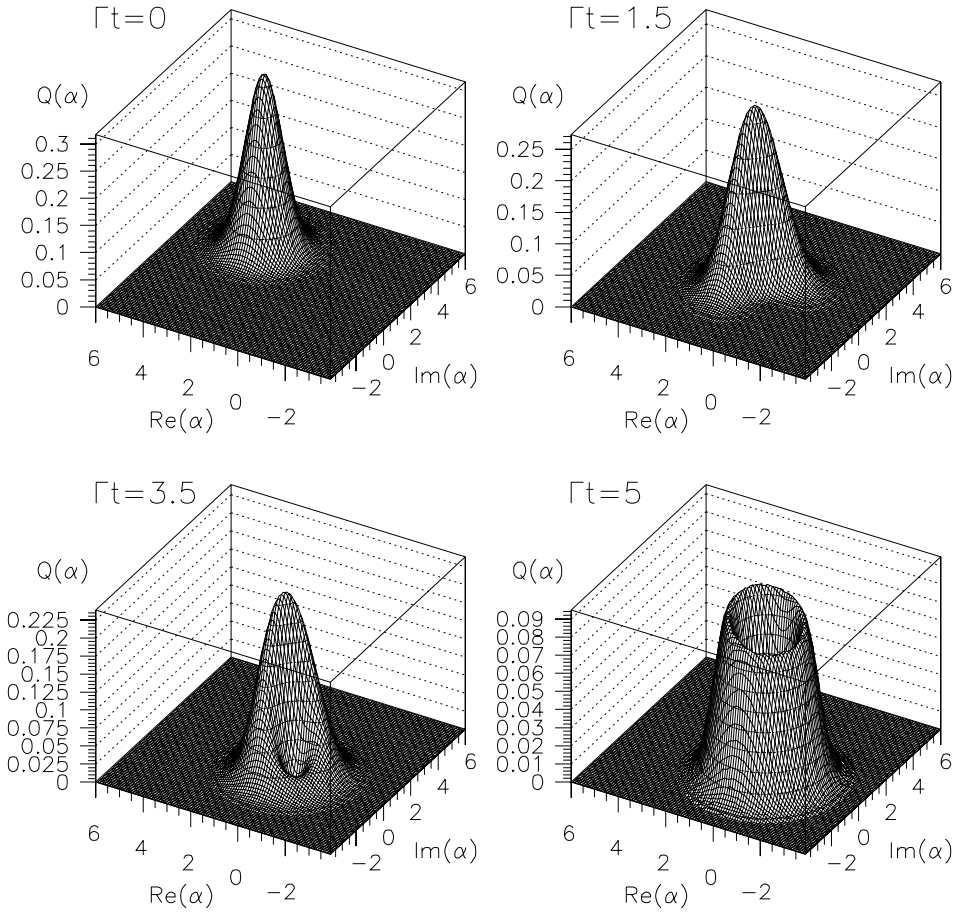

Figure 5. Evolution of the Q function, for the quantum jump approach, and an initial coherent state, with $\alpha_0 = \sqrt{15/2}(1 + \mathrm{i})$. The temperature of the reservoir corresponds to a number of thermal photons equal to two. The initial Gaussian, corresponding to a coherent state, evolves into the distribution corresponding to a jumping Fock state.

vacuum state and, from then on, the sequence of jumping Fock states.

Of course, this argument holds only if the photon-number distribution has a finite number of components in the Fock-state basis. Nevertheless, we show in section 5.3 that this property holds true in general, albeit in an average sense.

5.2. Diffusion-like evolution

We consider now the evolution corresponding to the situation displayed in figure 2. We consider as initial state the same coherent state as in figure 5, the reservoir temperature being also the same as before ($\overline{n} = 2$). In this case, the system evolves according to the HSSDE given by equation (54). After some time, the Q function approaches a distorted Gaussian, with a mild amount of squeezing along the direction of the axis corresponding to the real part of α (figure 6). The centre of this Gaussian keeps diffusing in phase space, so that after a long time span the time-averaged distribution coincides with the Bose–Einstein distribution. Similar localization patterns were demonstrated in [30, 31].

Again our model leads to a physical interpretation of this localization phenomenon: it is associated to the coherence transfer from the applied field to the cavity mode, through the interaction with the three-level atom.

5.3. Analytical proof of localization

For the quantum jump situation, it is actually possible to demonstrate analytically that the system evolves in the mean towards a Fock state, for non-zero temperatures.

We first define two kind of variances, for an arbitrary operator O.

For the Hermitian case:

$$\langle \Delta O^2 \rangle = \langle O^2 \rangle - \langle O \rangle^2, \tag{55}$$

and for the non-Hermitian case:

$$\begin{aligned} |\Delta O|^2 &= (O^\dagger - \langle O^\dagger \rangle)(O - \langle O \rangle) \\ &= O^\dagger O - \langle O^\dagger \rangle O - O^\dagger \langle O \rangle - \langle O^\dagger \rangle \langle O \rangle, \end{aligned} \tag{56}$$

so that

$$\langle |\Delta O|^2 \rangle = \langle O^\dagger O \rangle - \langle O^\dagger \rangle \langle O \rangle. \tag{57}$$

In particular,we are interested in two quantities:

$$Q_1 = \langle |\Delta a|^2 \rangle, \tag{58}$$

$$Q_2 = \langle |\Delta n|^2 \rangle, \tag{59}$$

that measure the distance of the state from being a coherent or a Fock state, respectively.

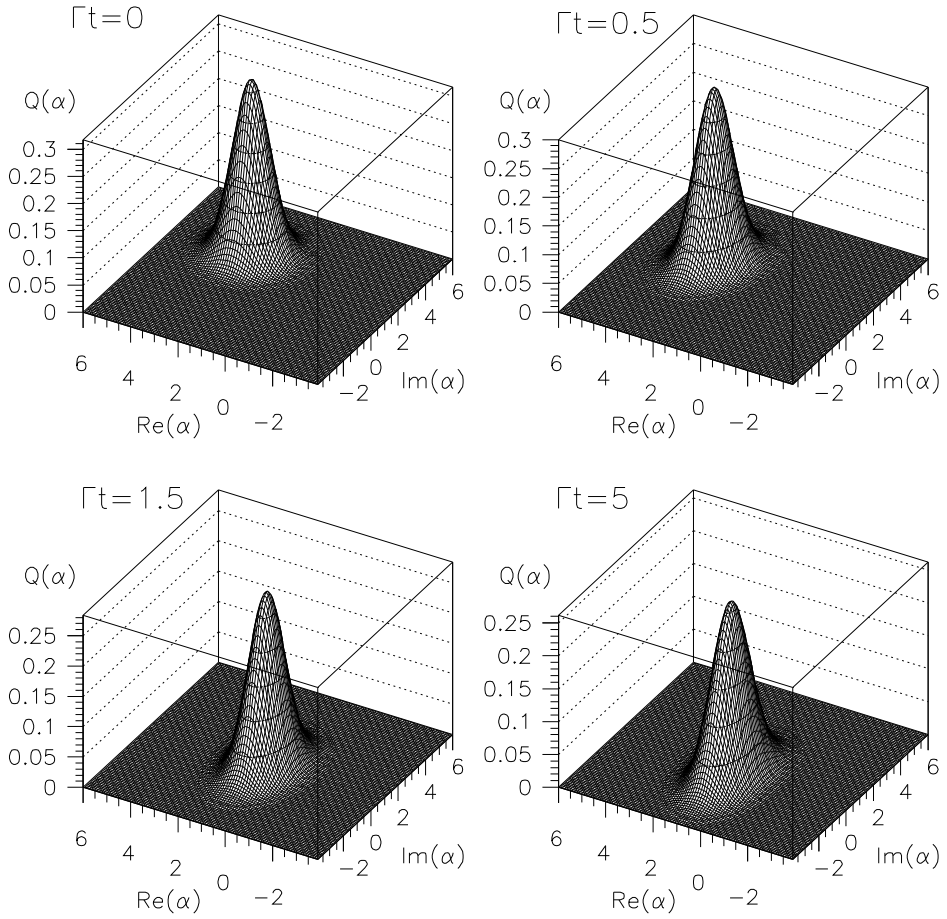

Figure 6. Evolution of the Q function, for the diffusive evolution, and an initial coherent state, with $\alpha_0 = \sqrt{15/2}(1+\mathrm{i})$. The temperature of the reservoir is the same as in figure 5. The initial Gaussian, corresponding to a coherent state, evolves into a distorted Gaussian, whose centre diffuses in phase space.

We start with the quantum jump equation:

$$|\mathrm{d}\psi\rangle = -\frac{\mathrm{i}}{\hbar}H|\psi\rangle\mathrm{d}t - \tfrac{1}{2}\sum_m (C_m^\dagger C_m - \langle C_m^\dagger\rangle\langle C_m\rangle)|\psi\rangle\mathrm{d}t + \sum_m \left(\frac{C_m}{\sqrt{C_m^\dagger C_m}} - 1\right)|\psi\rangle\delta N_m, \tag{60}$$

with:

$$M(\delta N_m) = \langle C_m^\dagger C_m\rangle\mathrm{d}t, \tag{61}$$

$$\delta N_m \delta N_m = \delta N_n \delta_{n,m}. \tag{62}$$

We will calculate, using Ito's rule of calculus, Q_1 and Q_2 for $T = 0$ ($C = \sqrt{\Gamma}a$) and $T > 0$ ($C_1 = \sqrt{(\bar{n}+1)\Gamma}a$, $C_2 = \sqrt{\Gamma\bar{n}}a^\dagger$).

We first develop some general expressions, which will be applied to calculate the above variances.

$$\mathrm{d}\langle O\rangle = \langle \mathrm{d}\psi|O|\psi\rangle + \langle\psi|O|\mathrm{d}\psi\rangle + \langle \mathrm{d}\psi|O|\mathrm{d}\psi\rangle = -\frac{\mathrm{i}}{\hbar}\langle[O,H]\rangle\mathrm{d}t - \tfrac{1}{2}\langle\{O,C^\dagger C\}\rangle\mathrm{d}t + \langle O\rangle\langle C^\dagger C\rangle\mathrm{d}t + \frac{(\langle C^\dagger OC\rangle - \langle C^\dagger C\rangle\langle O\rangle)}{\langle C^\dagger C\rangle}\delta N, \tag{63}$$

and similarly for the case in which several jump operators are present.

For the variance of a non-Hermitian operator, we have:

$$\mathrm{d}(\langle|\Delta O|^2\rangle) = \mathrm{d}\langle O^\dagger O\rangle - \langle O\rangle\mathrm{d}\langle O^\dagger\rangle - \langle O^\dagger\rangle\mathrm{d}\langle O\rangle - \mathrm{d}\langle O^\dagger\rangle\mathrm{d}\langle O\rangle. \tag{64}$$

After a simple calculation, one gets:

$$\mathrm{d}(\langle|\Delta O|^2\rangle) = -\frac{\mathrm{i}}{\hbar}\langle[|\Delta O|^2, H]\rangle\mathrm{d}t - \tfrac{1}{2}\langle\{|\Delta O|^2, C^\dagger C\}\rangle\mathrm{d}t + \langle|\Delta O|^2\rangle\langle C^\dagger C\rangle\mathrm{d}t - \langle|\Delta O|^2\rangle\delta N + \frac{\langle C^\dagger O^\dagger OC\rangle\langle C^\dagger C\rangle - \langle C^\dagger O^\dagger C\rangle\langle C^\dagger OC\rangle}{\langle C^\dagger C\rangle\langle C^\dagger C\rangle}\delta N. \tag{65}$$

In the Hermitian case, on the other hand, we get:

$$\mathrm{d}(\langle\Delta O^2\rangle) = -\frac{\mathrm{i}}{\hbar}\langle[\Delta O^2, H]\rangle\mathrm{d}t - \tfrac{1}{2}\langle\{\Delta O^2, C^\dagger C\}\rangle\mathrm{d}t + \langle\Delta O^2\rangle\langle C^\dagger C\rangle\mathrm{d}t - \langle\Delta O^2\rangle\delta N + \frac{\langle C^\dagger O^2 C\rangle\langle C^\dagger C\rangle - \langle C^\dagger OC\rangle\langle C^\dagger OC\rangle}{\langle C^\dagger C\rangle\langle C^\dagger C\rangle}\delta N. \tag{66}$$

Now we specialize to several cases:

(a) $T = 0$, $O = a$, $C = \sqrt{\Gamma}a$, and $H = \hbar\omega a^\dagger a$.

Using the above general expressions, we write:

$$\mathrm{d}(\langle|\Delta a|^2\rangle) = \Big[- \Gamma\langle a^\dagger a a^\dagger a\rangle - 2\Gamma\langle a^\dagger a\rangle\langle a^\dagger\rangle\langle a\rangle$$

$$+\Gamma\langle a^\dagger a\rangle\langle a^\dagger a\rangle + \frac{\Gamma}{2}\langle a^\dagger a^\dagger a\rangle\langle a\rangle + \frac{\Gamma}{2}\langle a^\dagger a a^\dagger\rangle\langle a\rangle$$
$$+\frac{\Gamma}{2}\langle a a^\dagger a\rangle\langle a^\dagger\rangle + \frac{\Gamma}{2}\langle a^\dagger a a\rangle\langle a^\dagger\rangle\Big]\,\mathrm{d}t$$
$$-\langle a^\dagger a\rangle\delta N + \langle a^\dagger\rangle\langle a\rangle\delta N$$
$$+\frac{\langle a^\dagger a^\dagger a a\rangle\langle a^\dagger a\rangle - \langle a^\dagger a^\dagger a\rangle\langle a^\dagger a a\rangle}{\langle a^\dagger a\rangle\langle a^\dagger a\rangle}\delta N. \tag{67}$$

The above results are neither strictly positive or negative, so we cannot draw any conclusion; however, for the statistical mean:

$$M\frac{\mathrm{d}(\langle|\Delta a|^2\rangle)}{\mathrm{d}t} = -\Gamma\langle|\Delta a|^{\langle 2\rangle}$$
$$-\frac{\Gamma\langle(\Delta a^\dagger)a^\dagger a\rangle\langle a^\dagger a\,\Delta a\rangle}{\langle a^\dagger a\rangle} \leqslant 0, \tag{68}$$

so, in the mean, the system goes to a coherent state, which, in this case, is the vacuum.

(b) $T > 0$, $O = a$, $C_1 = \sqrt{(\overline{n}+1)\Gamma}a$, $C_2 = \sqrt{\Gamma\overline{n}}a^\dagger$, and $H = \hbar\omega a^\dagger a$.

The reader can easily verify, with a little algebra, that, in this case, neither $\mathrm{d}(\langle|\Delta a|^2\rangle)$ or $M\mathrm{d}(\langle|\Delta a|^2\rangle)$ are strictly negative.

(c) $T > 0$, $O = a^\dagger a$, $C_1 = \sqrt{(\overline{n}+1)\Gamma}a$, $C_2 = \sqrt{\Gamma\overline{n}}a^\dagger$, and $H = \hbar\omega a^\dagger a$.

In this case, as shown in appendix B, $\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle$ is not negative, but $M\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle$ is:

$$M\frac{\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle}{\mathrm{d}t} = -\Gamma(\overline{n}+1)\frac{\langle(\Delta a^\dagger a)a^\dagger a\rangle\langle a^\dagger a(\Delta a^\dagger a)\rangle}{\langle a^\dagger a\rangle}$$
$$-\Gamma(\overline{n})\frac{\langle(\Delta a a^\dagger)a a^\dagger\rangle\langle a a^\dagger(\Delta a a^\dagger)\rangle}{\langle a a^\dagger\rangle} \leqslant 0. \tag{69}$$

So Q_2 is strictly diminishing in the mean, even at $T > 0$. Since Q_1 is not, the final state will not necessarily be the vacuum. Indeed, there is no unique final state in this case. It is easy to show from equation (69) that $M[\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle/\mathrm{d}t] = 0$ if and only if the state of the system is a Fock state. This result shows therefore that any initial state approaches eventually a Fock state $|n\rangle$, with n fluctuating with a mean $\overline{n}$. While the exact relation between the ensemble average behaviour and the long-time behaviour of a single trajectory is not completely obvious, it is clear that the probability of a trajectory violating these inequalities over a long period is very small. Once the trajectory approaches a Fock state, it remains a Fock state for all time thereafter. This is reflected in the numerical results.

6. Conclusions

We propose here a physical interpretation of the quantum jump approach and the HSSDE, using as an example the damping of one field mode in a cavity at temperature T.

This field damping mechanism can be modelled as an atomic beam, whose upper and lower population ratio is given by the Boltzmann factor, crossing a lossless cavity.

The quantum jump trajectory can be interpreted as a continuous monitoring of the outgoing two-level atoms, which are resonant with the cavity mode. We show both numerically and analytically that this continuous measurement on the reservoir leads, for each trajectory, to a pure Fock state. At a later time and due to the non-zero temperature, a thermal photon may produce a jump to a different Fock state, thus leading, as time goes on, to a series of Fock states, whose statistics will reproduce the thermal distribution.

In the case of the HSSDE, the proposed damping mechanism consists of a three-level atomic beam, with a split ground state, whose population ratio of the upper and lower levels is given by the Boltzmann factor. The atoms again cross a lossless cavity, being resonant with the mode of the field under consideration. A second field is externally applied, with the same frequency but different polarization, so that each of the two fields connects the upper atomic state with a different lower sub-level. If this external field is a strong classical field, we show analytically that the stochastic Schrödinger equation describing the behaviour of the quantum field in the cavity corresponds precisely to the HSSDE.

The beam is continuously monitored as it exits the cavity. Numerically, one observes, for low temperatures, that the state of the field goes to a mildly squeezed state, centred around a value of α which diffuses in phase space, in such a way that the time-averaged distribution again reproduces the thermal state.

Other kinds of stochastic equations can also be modelled by slightly modified detection schemes. For instance, a stochastic Schödinger equation of the heterodyne kind [11, 12] is obtained if the three-level atom is off-resonance with respect to the cavity and the applied field.

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Appendix A. Derivation of the HSSDE

Here we present the detailed derivation of the HSSDE.

We start from the expansion given by equation (11), which in the two-jump situation, and neglecting the commutators between the jump operators and the no-jump evolution (for the same reason as discussed in the previous section), can be expressed as:

$$\rho(\Delta t) = \sum_{m_1,m_2=0}^{\infty}\frac{(\Delta t)^{m_1+m_2}}{m_1!m_2!}S(\Delta t)J_2^{m_2}J_1^{m_1}\rho(0). \tag{A1}$$

The probability of m_1 and m_2 quantum jumps of the respective types, is given by:

$$P_{m_1,m_2}(\Delta t) = \frac{(\Delta t)^{m_1+m_2}}{m_1!m_2!}\,\mathrm{Tr}\{S(\Delta t)J_2^{m_2}J_1^{m_1}\rho(0)\}. \tag{A2}$$

The master equation of the field, corresponding to a lossless cavity at temperature T, may be written as:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = (J_1 + J_2)\rho - \frac{\Gamma}{2}\rho[a^\dagger a(1+2\overline{n}) + 2\varepsilon(1+\overline{n})a^\dagger + 2\varepsilon\overline{n}a. + \overline{n} + \varepsilon^2(1+2\overline{n})] - \frac{\Gamma}{2}[a^\dagger a(1+2\overline{n}) + 2\varepsilon(1+\overline{n})a + 2\varepsilon\overline{n}a^\dagger + \overline{n} + \varepsilon^2(1+2\overline{n})]\rho. \tag{A3}$$

Therefore, according to the discussion in section 2, one possible way of writing $S(\Delta t)$ is:

$$S(\Delta t)\rho = N(\Delta t)\rho N(\Delta t)^\dagger, \tag{A4}$$

with:

$$N(\Delta t) = \exp\left\{-\frac{\Gamma(\Delta t)}{2}[a^\dagger a(1+2\overline{n}) + 2\varepsilon(1+\overline{n})a^\dagger + 2\varepsilon\overline{n}a + \overline{n} + \varepsilon^2(1+2\overline{n})]\right\}. \tag{A5}$$

Using equations (A2) and (A5), we can write:

$$P_{m_1,m_2}(\Delta t) = \left[\frac{\exp\mu_1(\mu_1)^{m_1}}{m_1!}\right]\left[\frac{\exp\mu_2(\mu_2)^{m_2}}{m_2!}\right] \times \mathrm{Tr}\left[\exp(\beta')\left(1+\frac{a^\dagger}{\varepsilon}\right)^{m_2}\left(1+\frac{a}{\varepsilon}\right)^{m_1}\rho\left(1+\frac{a^\dagger}{\varepsilon}\right)^{m_1} \times \left(1+\frac{a}{\varepsilon}\right)^{m_2}\exp(\beta^{\dagger\prime})\right], \tag{A6}$$

where:

$$\begin{aligned}\mu_1 &= \Gamma\Delta t\varepsilon^2(1+\overline{n}),\\ \mu_2 &= \Gamma\Delta t\varepsilon^2\overline{n},\\ \beta' &= -\frac{-\Gamma\Delta t}{2}\{a^\dagger a(1+2\overline{n}) + 2[\varepsilon(1+\overline{n})a + \varepsilon\overline{n}a^\dagger] + \overline{n}\}.\end{aligned} \tag{A7}$$

From equation (A6), we can now calculate $\langle m_i\rangle$ and $\sigma_i^2 = \langle m_i^2\rangle - \langle m_i\rangle^2$ up to order $(\frac{1}{\varepsilon})^{\frac{3}{2}}$. The result is:

$$\begin{aligned}\langle m_i\rangle &= \mu_i(1+\frac{2}{\varepsilon}\langle X_1\rangle,\\ \sigma_i^2 &= \mu_i.\end{aligned} \tag{A8}$$

Now, we turn to the final step of this calculation, which yields the time evolution of the state vector.

After repeated jumps and no-jump events, the unnormalized wavefunction for the field can be written as:

$$|\tilde{\psi}\rangle_f(\Delta t) = N(\Delta t - t_m)C_2N(t_m - t_{m-1})C_1N\ldots|\psi\rangle_f(0),$$

or, except for an overall phase factor:

$$|\tilde{\psi}\rangle_f(\Delta t) = N(\Delta t)C_2^{m_2}C_1^{m_1}|\psi\rangle_f(0), \tag{A9}$$

where the *tilde* ($\sim$) indicates that the state vector is not normalized.

Using equations (A5) and (A9), one can write, up to a normalization constant:

$$|\tilde{\psi}\rangle_f(\Delta t) = \exp\left(-\frac{\Gamma(\Delta t)}{2}\{a^\dagger a(1+\overline{n}) + 2[\varepsilon(1+\overline{n})a^\dagger + \varepsilon\overline{n}a]\}\right) \times \left(1+\frac{a^\dagger}{\varepsilon}\right)^{m_2}\left(1+\frac{a}{\varepsilon}\right)^{m_1}|\psi\rangle_f(0), \tag{A10}$$

or, expanding, up to $\varepsilon^{-3/2}$:

$$|\tilde{\psi}\rangle_f(\Delta t) = \left[1 - \frac{\Gamma\Delta t}{2}(a^\dagger a(1+\overline{n}) + aa^\dagger\overline{n}) - \Gamma\Delta t\varepsilon(a(1+\overline{n}) + a^\dagger\overline{n})\right] \times \left[1+\frac{1}{\varepsilon}(m_1a + m_2a^\dagger)\right]|\psi\rangle_f(0). \tag{A11}$$

We are interested in the $\varepsilon \to \infty$ limit. In deriving equation (A11) we considered ε large, $\Gamma\Delta t \sim \varepsilon^{-3/2}$, and $m_1, m_2, \mu_1, \mu_2 \sim \varepsilon^{1/2}$.

Now, we consider two random numbers with non-zero average m_1 and m_2:

$$\begin{aligned}m_1 &= \langle m_1\rangle + \frac{\sigma_1}{\sqrt{\Delta t}}\Delta W_1,\\ m_2 &= \langle m_2\rangle + \frac{\sigma_2}{\sqrt{\Delta t}}\Delta W_2,\end{aligned} \tag{A12}$$

which satisfy:

$$\langle(\Delta W_1)^2\rangle = \langle(\Delta W_2)^2\rangle = \Delta t. \tag{A13}$$

We notice that ΔW_i are two independent Wiener processes.

Finally, equation (A11) can be written as:

$$\Delta^{m_1,m_2}|\tilde{\psi}\rangle_f(\Delta t) = |\tilde{\psi}\rangle_f(\Delta t) - |\psi\rangle_f(0) = \left\{\left[-\frac{\Gamma}{2}(1+\overline{n})a^\dagger a - \frac{\Gamma}{2}(\overline{n})aa^\dagger + 2\Gamma\langle X_1\rangle(a(1+\overline{n}) + a^\dagger\overline{n})\right]\Delta t + a^\dagger\sqrt{\Gamma\overline{n}}\Delta W_2 + a\sqrt{\Gamma(1+\overline{n})}\Delta W_1\right\}|\psi\rangle_f(0). \tag{A14}$$

which is the desired result.

Appendix B. Fluctuations

We want to calculate $\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle$ and $M\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle$.

We do it first in a simple case $T = 0$, $O = a^\dagger a$, $C = \sqrt{\Gamma}a$, and $H = \hbar\omega a^\dagger a$.

$$\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle = \Gamma\delta t\{-\langle a^\dagger aa^\dagger aa^\dagger a\rangle + 2\langle a^\dagger aa^\dagger a\rangle\langle a^\dagger a\rangle - 2\langle a^\dagger a\rangle\langle a^\dagger a\rangle\langle a^\dagger a\rangle + \langle a^\dagger aa^\dagger a\rangle\langle a^\dagger a\rangle\} - \langle a^\dagger aa^\dagger a\rangle\delta N + \langle a^\dagger a\rangle\langle a^\dagger a\rangle\delta N + \frac{\langle a^\dagger a^\dagger aa^\dagger aa\rangle\langle a^\dagger a\rangle - \langle a^\dagger a^\dagger aa\rangle\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle\langle a^\dagger a\rangle}\delta N, \tag{B1}$$

or:

$$\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle = -\Gamma\delta t\langle(\Delta a^\dagger a)(\Delta a^\dagger a)(\Delta a^\dagger a)\rangle - \langle(\Delta a^\dagger a)^2\rangle\delta N + \frac{\langle a^\dagger a^\dagger aa^\dagger aa\rangle\langle a^\dagger a\rangle - \langle a^\dagger a^\dagger aa\rangle\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle\langle a^\dagger a\rangle}\delta N. \tag{B2}$$

Now, we apply the above results to the more interesting case $T > 0$, $O = a^\dagger a$, $C_1 = \sqrt{(\overline{n}+1)\Gamma}a$, $C_2 = \sqrt{\Gamma\overline{n}}a^\dagger$, $H = \hbar\omega a^\dagger a$:

$$\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle = -\Gamma(\overline{n}+1)\langle((\Delta a^\dagger a))(\Delta a^\dagger a)(\Delta a^\dagger a)\rangle\mathrm{d}t - \langle(\Delta a^\dagger a)^2\rangle\delta N_1$$

$$+\frac{(\langle a^\dagger aa^\dagger aa^\dagger a\rangle\langle a^\dagger a\rangle - \langle a^\dagger aa^\dagger a\rangle\langle a^\dagger aa^\dagger a\rangle)\delta N_1}{\langle a^+a\rangle\langle a^\dagger a\rangle}$$
$$+\Gamma\bar{n}\mathrm{d}t[-\langle aa^\dagger aa^\dagger aa\dagger\rangle + 2\langle aa^\dagger aa^\dagger\rangle - \langle aa^\dagger\rangle$$
$$+2\langle aa^\dagger aa^\dagger\rangle\langle a^\dagger a\rangle - 2\langle aa^\dagger\rangle\langle a^\dagger a\rangle - \langle aa^\dagger\rangle\langle a^\dagger a\rangle\langle a^\dagger a\rangle$$
$$+\langle a^\dagger aa^\dagger a\rangle\langle aa^\dagger\rangle - \langle a^\dagger a\rangle\langle a^\dagger a\rangle\langle aa^\dagger\rangle]$$
$$-\langle(\Delta a^\dagger a)^2\rangle\delta N_2$$
$$+\frac{(\langle aa^\dagger aa^\dagger aa^\dagger\rangle\langle aa^\dagger\rangle - \langle aa^\dagger aa^\dagger\rangle\langle aa^\dagger aa^\dagger\rangle)\delta N_2}{\langle aa^+\rangle\langle aa^\dagger\rangle}. \quad \text{(B3)}$$

In the above expression, neither the deterministic or the stochastic term is definitely non-increasing. But in the mean it does decrease:

$$M\frac{\mathrm{d}\langle(\Delta a^\dagger a)^2\rangle}{\mathrm{d}t} = -\Gamma(\bar{n}+1)\frac{\langle(\Delta a^\dagger a)a^\dagger a\rangle\langle a^\dagger a(\Delta a^\dagger a)\rangle}{\langle a^+a\rangle}$$
$$-\Gamma\bar{n}\frac{\langle(\Delta aa^\dagger)aa^\dagger\rangle\langle aa^\dagger(\Delta aa^\dagger)\rangle}{\langle aa^+\rangle} \leqslant 0. \quad \text{(B4)}$$

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