Role of pumping statistics in laser dynamics: Quantum Langevin approach

Claus Benkert and M. O. Scully
Joint Institute for Laboratory Astrophysics, University of Colorado, Boulder, Colorado 80309-0440

J. Bergou,* L. Davidovich,† M. Hillery,‡ and M. Orszag§
Max-Planck-Institut für Quantenoptik, D-8046 Garching, Federal Republic of Germany and Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
(Received 28 July 1989)

We study in detail the influence of pumping statistics on the laser dynamics. We apply the technique of quantum Langevin operators and generalize the corresponding noise operators to incorporate the statistical properties of the pump mechanism. These equations are then used to derive expressions for the phase and intensity fluctuations of lasers with various pump statistics. We find that a reduction of pump noise can lead to a significant squeezing of the photon number noise below the shot-noise limit. This is in complete agreement with our previous analysis, which used a density-operator approach.

I. INTRODUCTION

Noise-reduction schemes in lasers and masers have played an important role in quantum optics during the last decade. Important examples are squeezed states of the radiation field[1] and correlated emission laser[2] (CEL) schemes. In most of these systems the emphasis was put on altering the characteristics of the quantum noise that is generated by spontaneous-emission events. In the present work the chief purpose is to study in depth the effects of the various pumping schemes and their fluctuations on the noise characteristics of the outcoming radiation.

In our previous paper,[3] we analyzed the influence of the pump statistics on the amplitude and phase fluctuations of the laser radiation through a density operator analysis. A generalized Scully-Lamb master equation was derived, in terms of a parameter p, in which p represented the probability of an atom to be excited to the upper state before entering the radiation cavity. The two extreme cases were $p \to 0$ (Poisson statistics) and $p = 1$ (regular injection). We found that the pump statistics have no influence on the phase fluctuations but can have a drastic effect on the photon number fluctuations. In the high-intensity limit the photon number noise can be suppressed below the shot-noise limit, producing squeezed states of the radiation field.

In the present paper we discuss the influence of the statistical properties of the pump mechanism on the electromagnetic field from a different point of view. We apply the formalism of Langevin operators and generalize the familiar noise operator to include the effect of pump noise. This allows us to analyze from first principles the effect of pump fluctuations on the radiated laser light. We demonstrate that well above threshold the photon number fluctuations can be squeezed below the shot-noise limit, provided the noise of the pump source is smaller than that of a Poisson statistic. Furthermore, we show that the possible amount of squeezing in a pump-noise suppressed laser depends on the decay constants of the lasing atomic levels. This generalizes the discussion of our previous paper[3] and enables an easy comparison with the results of other authors.[4]

In Sec. II we derive the Langevin operator equations for the field and atomic variables. Furthermore, we calculate the correlation functions for the corresponding noise operators. These noise operators now incorporate the effects of two different noise sources: the usual fluctuations due to the coupling to damping reservoirs and the influence of fluctuations of the pump mechanism.

In Sec. III, we convert the operator Langevin equations into c-number stochastic differential equations. We then assume that the atomic time scale is much shorter than the time scale of the radiation field. This allows us to eliminate the atomic variables adiabatically so that we obtain an equation for the electromagnetic field alone. The calculated diffusion coefficients for the field then completely specify the noise properties of the radiated light.

In Sec. IV we discuss the implications of our results on the phase and intensity of the electromagnetic field. In agreement with our previous density-operator analysis, we find that the phase diffusion is unaffected by the fluctuations of the pump mechanism. However, the photon number noise crucially depends on the pump fluctuations. In the case of equal decay constants of the lasing atomic levels we find up to 25% squeezing of the photon number noise, provided the atoms are injected regularly ($p = 1$). If the decay constant of the lower level is much larger than the one of the upper excited level, squeezing of up to 50% is possible. Such a situation is of particular interest for semiconductor lasers and the results are in agreement with treatments by other authors.[5]

II. QUANTUM LANGEVIN EQUATIONS

In our physical model of a laser with various pump fluctuations, a beam of three-level atoms is injected into a laser cavity (see Fig. 1). The atoms are initially prepared in the upper level $|a\rangle$. The two levels $|a\rangle$ and $|b\rangle$ con-

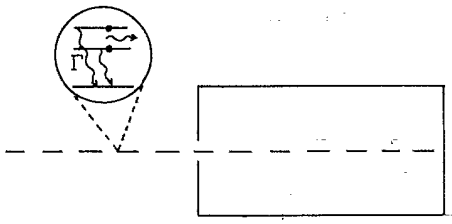

FIG. 1. Physical model of a laser with arbitrary pump fluctuations. Three-level atoms, which are initially in the upper excited state, are injected into the laser cavity. The specific way of atomic injection models various types of pump fluctuations.

stitute the lasing transition which is coupled to one mode of the radiation field inside the cavity. The lowest atomic level $|c\rangle$ is an inert ground state to which the atoms decay with a decay rate Γ. Note that no assumption is made for the particular way of atomic injection. In fact, the statistical distribution over the injection times of individual atoms determines the fluctuations of the pump. This enables us to discuss different cases of pump fluctuations within a unified framework. The Hamiltonian of such a system in the rotating wave approximation is given by

$$H=\hbar\omega a^{\dagger}a+\sum_j(\epsilon_a|a\rangle\langle a|+\epsilon_b|b\rangle\langle b|+\epsilon_c|c\rangle\langle c|)_j +\hbar g\sum_j\Theta(t-t_j)(a^{\dagger}\sigma^j+\sigma^{j\dagger}a)\ . \tag{1}$$

Here a and $a^{\dagger}$ are the creation and annihilation operator for the electromagnetic field, while σ^j is the atomic polarization operator $(|b\rangle\langle a|)_j$. The parameter g denotes the coupling constant between atoms and field and $\Theta(t)$ is the step function. The cavity losses and the atomic decay are modeled in the standard[6,7] way by coupling the radiation field and each atom to heat reservoirs. We then find the following quantum Langevin equations for the field and atomic operators:

$$\dot{a}=-i\omega a-\frac{\gamma}{2}a-ig\sum_j\Theta(t-t_j)\sigma^j+F_{\gamma}\ , \tag{2}$$

$$\dot{\sigma}^j=-i\omega_{ab}\sigma^j-\Gamma\sigma^j+ig\Theta(t-t_j)(\sigma^j_{aa}-\sigma^j_{bb})a+F^j_{ba}\ , \tag{3}$$

$$\dot{\sigma}^j_{aa}=-\Gamma\sigma^j_{aa}+ig\Theta(t-t_j)(a^{\dagger}\sigma^j-\sigma^{j\dagger}a)+F^j_{aa}\ , \tag{4}$$

$$\dot{\sigma}^j_{bb}=-\Gamma\sigma^j_{bb}-ig\Theta(t-t_j)(a^{\dagger}\sigma^j-\sigma^{j\dagger}a)+F^j_{bb}\ , \tag{5}$$

in which $\sigma^j_{aa}=(|a\rangle\langle a|)_j$, $\sigma^j_{bb}=(|b\rangle\langle b|)_j$, and $\omega_{ab}=(\epsilon_a-\epsilon_b)/\hbar$. For simplicity we have assumed all atomic decay constants to be equal. This assumption will be relaxed later in Sec. IV. The operators F on the right-hand side of Eqs. (2)–(5) are the typical Langevin noise operators which arise through the interaction with a heat bath. These operators are specified through their first and second moment. The normally ordered noise correlation functions for F_{γ} are found to be[6]

$$\langle F_{\gamma}(t)\rangle=0\ , \tag{6a}$$

$$\langle F^{\dagger}_{\gamma}(t)F_{\gamma}(t')\rangle=n_{\rm th}\delta(t-t')\ ,\quad \langle F_{\gamma}(t)F_{\gamma}(t')\rangle=0\ , \tag{6b}$$

in which $n_{\rm th}$ is the number of thermal photons in the cavity. For simplicity we assume that the heat reservoir is at zero temperatures so that $n_{\rm th}$ is equal to zero.

The moments of the atomic noise operators are discussed in detail by several authors.[6,7] If the reservoirs only induce a decay from the atomic levels a and b to the ground state c, these moments are particularly simple. Some of the nonvanishing moments of interest are given by[7]

$$\langle F^{j\dagger}_{ba}(t)F^j_{ba}(t')\rangle=\Gamma\langle\sigma^j_{aa}(t)\rangle\delta(t-t')\ ,$$
$$\langle F^{j\dagger}_{ll}(t)F^j_{ll}(t')\rangle=\Gamma\langle\sigma^j_{ll}(t)\rangle\delta(t-t'),\quad l=a,b$$
$$\langle F^j_{ba}(t)F^j_{aa}(t')\rangle=\Gamma\langle\sigma^j(t)\rangle\delta(t-t')\ ,$$
$$\langle F^{j\dagger}_{ba}(t)F^j_{bb}(t')\rangle=\Gamma\langle\sigma^{j\dagger}(t)\rangle\delta(t-t')\ . \tag{7}$$

We now eliminate the quickly time-varying part in Eqs. (2) to (5) by moving into a rotating frame. For simplicity we assume resonance between the cavity mode frequency and the atomic transition frequency, i.e., $\omega=\omega_{ab}$. We can then define the slowly varying operators

$$\tilde{a}(t)=e^{i\omega t}a(t),\quad \tilde{\sigma}^j(t)=e^{i\omega t}\sigma^j(t)\ . \tag{8}$$

The equations of motion for $\tilde{a}$ and $\tilde{\sigma}^j$ are the same as those for a and σ^j with the only difference that the terms proportional to ω and ω_{ab} are omitted. We will therefore drop the tilde on the operators in the following analysis, keeping in mind that the operators a and σ^j now denote the slowly varying operators in the rotating frame.

As a next step we change from the operators for individual atoms to operators which describe the macroscopic atomic properties. This proves to be necessary for the approximation techniques employed in Sec. III. While the individual atomic operators are very sensitive towards an adiabatic approximation, the averaged, macroscopic quantities can be treated by such a technique. Therefore we define the operators

$$M(t)=-i\sum_j\Theta(t-t_j)\sigma^j(t)\ , \tag{9}$$

$$N_a(t)=\sum_j\Theta(t-t_j)\sigma^j_{aa}(t)\ , \tag{10}$$

$$N_b(t)=\sum_j\Theta(t-t_j)\sigma^j_{bb}(t)\ . \tag{11}$$

The operator M represents the macroscopic atomic polarization. The factor $(-i)$ in Eq. (9) has been chosen because of mathematical convenience. Furthermore, N_a and N_b specify the number of atoms in the two excited atomic levels a and b, respectively. With these definitions Eq. (2) for the electromagnetic field simplifies to

$$\dot{a}=-\frac{\gamma}{2}a+gM+F_{\gamma}\ . \tag{12}$$

The Langevin equations for the atomic operators can

be found by differentiating Eqs. (9)–(11) and substituting Eqs. (3)–(5), respectively. For example, for the operator N_a we obtain

$$\dot{N}_a=\sum_j[\delta(t-t_j)\sigma_{aa}^j(t)+\Theta(t-t_j)\dot{\sigma}_{aa}^j(t)]$$
$$=\sum_j\delta(t-t_j)\sigma_{aa}^j(t_j)-\Gamma N_a-g(a^\dagger M+M^\dagger a)$$
$$-i\sum_j\Theta(t-t_j)F_{ba}^j(t)\ . \tag{13}$$

The first term on the right-hand side of Eq. (13) corresponds to a pumping of the atoms into their upper excited state. To see this most clearly let us calculate the expectation value of this term,

$$\left\langle\sum_j\delta(t-t_j)\sigma_{aa}^j(t_j)\right\rangle=\left\langle\sum_j\delta(t-t_j)\langle\sigma_{aa}^j(t_j)\rangle\right\rangle_S$$
$$=\left\langle\sum_j\delta(t-t_j)\right\rangle_S\ . \tag{14}$$

Here we have made use of the fact that the atoms are initially prepared in their upper atomic level so that $\langle\sigma_{aa}^j(t_j)\rangle=1$. The index S on the brackets in Eq. (14) indicates that we still have to perform the statistical average over the injection times, i.e., the average over the pump statistics. If we assume a mean, time-independent atomic injection rate R, this average can be calculated as

$$\left\langle\sum_j\delta(t-t_j)\right\rangle_S=R\int_{-\infty}^{\infty}dt_j\delta(t-t_j)$$
$$=R\ . \tag{15}$$

Alternatively, Eq. (15) can be regarded as the definition for the averaged atomic injection rate R.

In order to separate the drift terms from the noise terms in Eq. (13) we add and subtract the expectation value of the first term and obtain

$$\dot{N}_a=R-\Gamma N_a-g(a^\dagger M+M^\dagger a)+F_a\ , \tag{16}$$

with

$$F_a(t)=\sum_j\Theta(t-t_j)F_{aa}^j(t)+\sum_j\delta(t-t_j)\sigma_{aa}^j(t_j)-R\ . \tag{17}$$

The operator F_a is the total noise operator for the atomic quantity N_a. It incorporates the contributions from the reservoir-induced decay of the atoms and the influence of pump fluctuations. It is easy to verify that the expectation value of $F_a(t)$ is equal to zero at all times, as expected for a Langevin noise operator.

In a similar way we can derive the equations for the remaining atomic operators

$$\dot{N}_b=-\Gamma N_b+g(a^\dagger M+M^\dagger a)+F_b\ , \tag{18}$$

$$\dot{M}=-\Gamma M+g(N_a-N_b)a+F_M\ , \tag{19}$$

with

$$F_b(t)=\sum_j\Theta(t-t_j)F_{bb}^j(t)+\sum_j\delta(t-t_j)\sigma_{bb}^j(t_j)\ , \tag{20}$$

$$F_M(t)=-i\sum_j\Theta(t-t_j)F_{ba}^j(t)-i\sum_j\delta(t-t_j)\sigma^j(t_j)\ . \tag{21}$$

Note that there is no pumping term in the Eqs. (18) and (19) because we assumed the atoms to be initially in the excited state $|a\rangle$. A generalization of this assumption is straightforward and will be discussed elsewhere.

We now turn to the evaluation of the noise correlation functions. As an example, let us calculate the two-time correlation for F_a

$$\langle F_a(t)F_a(t')\rangle=\left\langle\sum_{j,k}\Theta(t-t_j)\Theta(t'-t_k)\langle F_{aa}^j(t)F_{aa}^k(t')\rangle\right\rangle_S+\left\langle\sum_{j,k}\delta(t-t_j)\delta(t'-t_k)\langle\sigma_{aa}^j(t_j)\sigma_{aa}^k(t_k)\rangle\right\rangle_S-R^2\rho_{aa}^2\ . \tag{22}$$

In Eq. (22), we again have to consider two different averages: the usual quantum average over the bath variables and a statistical average over the injection times. We explicitly indicate the latter by a subscript S. Furthermore, we denoted the expectation value $\langle\sigma_{aa}^j(t_j)\rangle$ by ρ_{aa} to emphasize the different kinds of terms in the following analysis. At the end we will set ρ_{aa} equal to 1 to be consistent with the initial preparation of the atoms.

For the evaluation of the different terms Eq. (22), we note that the individual atoms are completely independent of each other. Therefore only the terms with $j=k$ contribute to the first term in Eq. (22). Furthermore, in the second term we can separate the expectation value $\langle\sigma_{aa}^j(t_j)\sigma_{aa}^k(t_k)\rangle$ for $j\neq k$ into the product $\langle\sigma_{aa}^j(t_j)\rangle\langle\sigma_{aa}^k(t_k)\rangle=\rho_{aa}^2$. We then obtain

$$\langle F_a(t)F_a(t')\rangle=\left\langle\sum_j\Theta(t-t_j)\Gamma\langle\sigma_{aa}^j(t)\rangle\right\rangle_S\delta(t-t')+\left\langle\sum_j\delta(t-t_j)\delta(t'-t_j)\rho_{aa}\right\rangle_S+\left[\left\langle\sum_{j\neq k}\delta(t-t_j)\delta(t'-t_k)\right\rangle_S-R^2\right]\rho_{aa}^2\ , \tag{23}$$

which we can simplify to

$$\langle F_a(t)F_a(t')=\langle\Gamma N_a\rangle\delta(t-t')+\left\langle\sum_j\delta(t-t_j)\right\rangle_S\rho_{aa}\delta(t-t')+\left[\left\langle\sum_{j\neq k}\delta(t-t_j)\delta(t'-t_k)\right\rangle_S-R^2\right]\rho_{aa}^2\ . \tag{24}$$

In Appendix A we have shown from first principles that

$$\left[\left\langle\sum_{j\neq k}\delta(t-t_j)\delta(t'-t_k)\right\rangle_S-R^2\right]=-pR\delta(t-t')\ , \tag{25}$$

in which the parameter p is a measure of the amount of antibunching among the atoms. For a perfectly regular atomic injection we find $p=1$. For a Poissonian injection statistic, which reveals a strong bunching effect among

the atoms, the parameter is given by $p=0$. Note that this parameter corresponds to the parameter p in our previous density operator analysis.[3] Substituting Eqs. (15) and (25) into Eq. (24) we find our final result for the noise correlation function

$$\langle F_a(t)F_a(t')\rangle=[\langle \Gamma N_a\rangle+R(\rho_{aa}-p\rho_{aa}^2)]\delta(t-t')$$
$$=[\langle \Gamma N_a\rangle+R(1-p)]\delta(t-t') . \quad (26)$$

The remaining noise correlation functions can be calculated in an analogous way. Some nonvanishing correlation functions of interest are

$$\langle F_M^\dagger(t)F_M(t')\rangle=[\Gamma\langle N_a\rangle+R]\delta(t-t') ,$$
$$\langle F_b(t)F_b(t')\rangle=\langle \Gamma N_b\rangle\delta(t-t') , \quad (27)$$
$$\langle F_b(t)F_M(t')\rangle=\langle \Gamma M\rangle\delta(t-t') .$$

Equations (12), (16), (18), and (19) now constitute a complete set of coupled operator equations which completely describe the laser with arbitrary pump statistics.

III. CORRESPONDING C-NUMBER LANGEVIN EQUATIONS

Before solving the equations of motion for the four macroscopic quantities we first convert the operator Langevin equations into corresponding c-number equations. This simplifies the following analysis. In order to convert the operator equations we have to define a certain ordering of the operators, to which the c-number equations correspond. This is necessary because the c numbers commute with each other while the operators do not. Therefore we obtain a unique relationship between operator and c-number Langevin equations only if we define the correspondence between a product of c numbers and a product of operators. We here choose the normal ordering $a^\dagger, M^\dagger, N_a, N_b, M, a$, and can now derive four c-number Langevin equations for the variables $\mathcal{E}$, $\mathcal{M}$, $\mathcal{N}_a$, and $\mathcal{N}_b$ such that the equations for their first and second moments are identical. Equations (12), (16), (18), and (19) are already in chosen order so that we immediately obtain

$$\dot{\mathcal{E}}=-\frac{\gamma}{2}\mathcal{E}+g\mathcal{M}+\mathcal{F}_\gamma , \quad (28)$$
$$\dot{\mathcal{M}}=-\Gamma\mathcal{M}+g(\mathcal{N}_a-\mathcal{N}_b)\mathcal{E}+\mathcal{F}_\mathcal{M} , \quad (29)$$
$$\dot{\mathcal{N}}_a=R-\Gamma\mathcal{N}_a-g(\mathcal{E}^*\mathcal{M}+\mathcal{M}^*\mathcal{E})+\mathcal{F}_a , \quad (20)$$
$$\dot{\mathcal{N}}_b=-\Gamma\mathcal{N}_b+g(\mathcal{E}^*\mathcal{M}+\mathcal{M}^*\mathcal{E})+\mathcal{F}_b . \quad (31)$$

The functions $\mathcal{F}$ in Eqs. (28)–(31) are again the typical Langevin noise forces with the expectation values

$$\langle \mathcal{F}_k(t)\rangle=0 , \quad (32)$$
$$\langle \mathcal{F}_k(t)\mathcal{F}_l(t')\rangle=\langle 2D_{kl}\rangle\delta(t-t') , \quad (33)$$

in which $\mathcal{F}_k$ and $\mathcal{F}_l$ can be any of the above noise forces. The diffusion coefficients D_{kl} are now determined by the requirement that the equations of motion for the second moments are also identical to the corresponding operator equations. It is easy to see that the diffusion coefficients for the noise force $\mathcal{F}_\gamma$ are the same as for the normally ordered noise operator F_γ, so that

$$D_{\gamma^*\gamma}=0, \quad D_{\gamma\gamma}=0 . \quad (34)$$

However, some of the atomic diffusion coefficients change in the transition from operator to c-number equation. As an example, let us calculate the diffusion coefficient $D_{\mathcal{M}\mathcal{M}}$. From the operator equation (19), we obtain

$$\frac{d}{dt}\langle M(t)M(t)\rangle=-2\Gamma\langle MM\rangle$$
$$+g[\langle (N_a-N_b)Ma\rangle$$
$$+\langle M(N_a-N_b)a\rangle]$$
$$+\langle MF_M\rangle+\langle F_M M\rangle . \quad (35)$$

We note that the third term in the square brackets is not in our chosen order because the operator M is to the left of N_a and N_b. Therefore we have to use the commutation $[M,N_a-N_b]=2M$ to bring this term into chosen order. Also, the last two terms vanish so that we obtain

$$\frac{d}{dt}\langle M(t)M(t)\rangle=-2\Gamma\langle MM\rangle$$
$$+2g\langle (N_a-N_b)Ma\rangle+2g\langle Ma\rangle . \quad (36)$$

We now use Eq. (29) to obtain the corresponding c-number equation

$$\frac{d}{dt}\langle \mathcal{M}(t)\mathcal{M}(t)\rangle=-2\Gamma\langle \mathcal{M}\mathcal{M}\rangle+2g\langle (\mathcal{N}_a-\mathcal{N}_b)\mathcal{M}\mathcal{E}\rangle$$
$$+\langle 2D_{\mathcal{M}\mathcal{M}}\rangle . \quad (37)$$

If we require the left-hand sides of Eqs. (36) and (37) to be equal we see that the diffusion coefficient $D_{\mathcal{M}\mathcal{M}}$ is given by

$$2D_{\mathcal{M}\mathcal{M}}=2g\mathcal{M}\mathcal{E} . \quad (38)$$

The remaining nonvanishing c-number diffusion coefficients can be calculated in an analogous way and are summarized in Table I.

We are now in the position to solve the Langevin Eqs. (28)–(31). Typically, the atomic decay rate Γ is much larger than the photon decay rate γ, so that the evolution of the atomic variables happens on a much shorter time scale than the electromagnetic field. We can then adia-

TABLE I. c-number diffusion coefficients for the atomic polarization and population variables.

$2D_{\mathcal{M}^*\mathcal{M}}=\Gamma\mathcal{N}_a+R$
$2D_{\mathcal{M}\mathcal{M}}=2g\mathcal{M}\mathcal{E}+\text{c.c.}$
$2D_{\mathcal{N}_b\mathcal{M}}=\Gamma\mathcal{M}+\text{c.c.}$
$2D_{\mathcal{N}_a\mathcal{N}_a}=\Gamma\mathcal{N}_a+R(1-p)-g(\mathcal{E}^*\mathcal{M}+\mathcal{M}^*\mathcal{E})$
$2D_{\mathcal{N}_b\mathcal{N}_b}=\Gamma\mathcal{N}_b-g(\mathcal{E}^*\mathcal{M}+\mathcal{M}^*\mathcal{E})$
$2D_{\mathcal{N}_a\mathcal{N}_b}=g(\mathcal{E}^*\mathcal{M}+\mathcal{M}^*\mathcal{E})$

batically eliminate the atomic variables $\mathcal{M}$, $\mathcal{N}_a$, and $\mathcal{N}_b$ and derive an equation for the field $\mathcal{E}$ alone. Thus as a first step we set the time derivative of $\mathcal{M}$ in Eq. (29) equal to zero and obtain the adiabatic value for the atomic polarization,

$$\mathcal{M}=\frac{g}{\Gamma}(\mathcal{N}_a-\mathcal{N}_b)\mathcal{E}+\frac{1}{\Gamma}\mathcal{F}_{\mathcal{M}} . \tag{39}$$

Substituting this result into the equations for $\mathcal{E}$, $\mathcal{N}_a$, and $\mathcal{N}_b$ yields

$$\dot{\mathcal{E}}=-\frac{\gamma}{2}\mathcal{E}+\frac{g^2}{\Gamma}(\mathcal{N}_a-\mathcal{N}_b)\mathcal{E}+\mathcal{F}_\gamma+\frac{g}{\Gamma}\mathcal{F}_{\mathcal{M}} , \tag{40}$$

$$\dot{\mathcal{N}}_a=R-\Gamma\mathcal{N}_a-\frac{2g^2}{\Gamma}(\mathcal{N}_a-\mathcal{N}_b)\mathcal{E}^*\mathcal{E} -\frac{g}{\Gamma}(\mathcal{F}^*_{\mathcal{M}}\mathcal{E}+\mathcal{E}^*\mathcal{F}_{\mathcal{M}})+\mathcal{F}_a , \tag{41}$$

$$\dot{\mathcal{N}}_b=-\Gamma\mathcal{N}_b+\frac{2g^2}{\Gamma}(\mathcal{N}_a-\mathcal{N}_b)\mathcal{E}^*\mathcal{E} +\frac{g}{\Gamma}(\mathcal{F}^*_{\mathcal{M}}\mathcal{E}+\mathcal{E}^*\mathcal{F}_{\mathcal{M}})+\mathcal{F}_b , \tag{42}$$

We next adiabatically eliminate the population variables $\mathcal{N}_a$ and $\mathcal{N}_b$ by setting their time derivative equal to zero. Equations (41) and (42) then reduce to a set of two coupled linear equations which can be easily solved. The result is

$$\mathcal{N}_a=\left[R\left[1+\frac{2g^2}{\Gamma^2}I\right]+\left[1+\frac{2g^2}{\Gamma^2}I\right]\mathcal{G}_a +\frac{2g^2}{\Gamma^2}I\mathcal{G}_b\right]\Big/\Gamma\left[1+\frac{4g^2}{\Gamma^2}I\right] , \tag{43}$$

$$\mathcal{N}_b=\left[R\frac{2g^2}{\Gamma^2}I+\left[1+\frac{2g^2}{\Gamma^2}I\right]\mathcal{G}_b +\frac{2g^2}{\Gamma^2}I\mathcal{G}_a\right]\Big/\Gamma\left[1+\frac{4g^2}{\Gamma^2}I\right] , \tag{44}$$

in which I is the intensity $\mathcal{E}^*\mathcal{E}$ of the radiation field and the noise functions $\mathcal{G}_a$ and $\mathcal{G}_b$ are defined by

$$\mathcal{G}_a=\mathcal{F}_a-\frac{g}{\Gamma}(\mathcal{F}^*_{\mathcal{M}}\mathcal{E}+\mathcal{E}^*\mathcal{F}_{\mathcal{M}}) , \tag{45}$$

$$\mathcal{G}_b=\mathcal{F}_b+\frac{g}{\Gamma}(\mathcal{F}^*_{\mathcal{M}}\mathcal{E}+\mathcal{E}^*\mathcal{F}_{\mathcal{M}}) . \tag{46}$$

We can now substitute the expressions (43) and (44) into (40) and obtain an equation of motion for the electromagnetic field alone,

$$\dot{\mathcal{E}}=-\frac{\gamma}{2}\mathcal{E}+\frac{\alpha}{2}\left[\frac{1}{1+\beta/\alpha I}\right]\mathcal{E}+\mathcal{F}_{\mathcal{E}} , \tag{47}$$

in which the noise force $\mathcal{F}_{\mathcal{E}}$ is given by

$$\mathcal{F}_{\mathcal{E}}=\mathcal{F}_\gamma+\frac{g}{\Gamma}\mathcal{F}_{\mathcal{M}}+\frac{g^2}{\Gamma^2}\left[\frac{1}{1+\beta/\alpha I}\right](\mathcal{G}_a-\mathcal{G}_b)\mathcal{E} . \tag{48}$$

The parameters α and β are the well-known[6] gain and saturation coefficients for a laser. They are defined as

$$\alpha=\frac{2g^2R}{\Gamma^2},\quad \beta=\frac{8g^4R}{\Gamma^4} . \tag{49}$$

The noise force $\mathcal{F}_{\mathcal{E}}$ is characterized by the correlation functions

$$\langle\mathcal{F}_{\mathcal{E}}(t)\rangle=0 , \tag{50a}$$

$$\langle\mathcal{F}^*_{\mathcal{E}}(t)\mathcal{F}_{\mathcal{E}}(t')\rangle=\langle 2D_{\mathcal{E}^*\mathcal{E}}\rangle\delta(t-t') , \tag{50b}$$

$$\langle\mathcal{F}_{\mathcal{E}}(t)\mathcal{F}_{\mathcal{E}}(t')\rangle=\langle 2D_{\mathcal{E}\mathcal{E}}\rangle\delta(t-t') . \tag{50c}$$

The diffusion coefficients $D_{\mathcal{E}^*\mathcal{E}}$ and $D_{\mathcal{E}\mathcal{E}}$ determine the strength of the noise and can be calculated from the definition of $\mathcal{F}_{\mathcal{E}}$. A lengthy but straightforward calculation yields the results

$$2D_{\mathcal{E}^*\mathcal{E}}=\alpha\left[\frac{1}{1+\beta/\alpha I}\right]^2\left[1+\frac{\beta}{4\alpha}I\left[3-\frac{p}{2}+\frac{\beta}{\alpha}I\right]\right] \tag{51}$$

and

$$2D_{\mathcal{E}\mathcal{E}}=-\alpha\left[\frac{1}{1+\beta/\alpha I}\right]^2\frac{\beta}{4\alpha}\mathcal{E}^2\left[3+\frac{p}{2}+\frac{\beta}{\alpha}I\right] . \tag{52}$$

These results will now be used to discuss the steady-state operation and the fluctuations of a laser with a given pump statistic.

IV. DISCUSSION

In this section we want to analyze in detail the properties of the intensity and phase of the radiated light. For this purpose we change into a polar coordinate system by defining

$$\mathcal{E}=\sqrt{I}e^{i\varphi} . \tag{53}$$

From the equation of motion (47) for the radiation field $\mathcal{E}$ we can now derive Langevin equations for the variables I and φ

$$\dot{\varphi}=F_\varphi , \tag{54}$$

$$\dot{I}=-\gamma I+\frac{\alpha}{1+\beta/\alpha I}I+F_I . \tag{55}$$

In Eq. (55) we have neglected the noise-induced drift terms which are much smaller than any of the remaining contributions. The diffusion coefficients for the noise forces F_φ and F_I are found to be

$$D_{\varphi\varphi}=\frac{\alpha}{4I}\left[\frac{1}{1+\beta/\alpha I}\right]\left[1+\frac{\beta}{2\alpha}I\right] \tag{56}$$

and

$$D_{II}=\frac{\alpha}{(1+\beta/\alpha I)^2}\left[1-p\frac{\beta}{4\alpha}I\right]I . \tag{57}$$

Before we start our discussion of the above quantities φ and I, let us first analyze their relationship to the phase and photon number of the radiation field. As it is well

known, a Hermitian phase operator does not exist in a strict quantum-mechanical sense. However, if the photon number of the electromagnetic laser field is large, the phase of the classical quantity $\mathcal{E}$ in Eq. (53) is an excellent approximation of the measured phase quadrature of the field. We can therefore directly identify the variable φ with the phase of the radiation field. The relationship between the quantity I and the photon statistics of the laser requires a more careful analysis. This is due to the fact that the intensity I corresponds to normally ordered products of the operators of the field. It is easy to see that the photon number $\langle n \rangle$ is directly given by

$$\langle n \rangle = \langle a^\dagger a \rangle = \langle I \rangle \ . \tag{58}$$

For the photon number variance we find

$$\begin{aligned}\langle (\Delta n)^2 \rangle &= \langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle^2 \\ &= \langle a^\dagger a^\dagger a a \rangle - \langle a^\dagger a \rangle^2 + \langle a^\dagger a \rangle \\ &= \langle I^2 \rangle - \langle I \rangle^2 + \langle I \rangle \\ &= \langle (\Delta I)^2 \rangle + \langle I \rangle \ . \end{aligned} \tag{59}$$

We are now in the position to discuss the phase and the photon number of our laser model. The steady-state intensity of the laser is found by setting the drift in Eq. (55) equal to zero,

$$I_0 = \frac{\alpha}{\beta}\frac{\alpha-\gamma}{\gamma} \ . \tag{60}$$

In contrast, the phase of the electromagnetic field is not locked to a particular value but can freely diffuse over the whole angle of 2π. The rate of phase diffusion is calculated with the help of Eq. (54)

$$\begin{aligned}\frac{d}{dt}\langle \varphi^2 \rangle &= \frac{d}{dt}\int_0^t dt' \int_0^t dt'' \langle \dot{\varphi}(t')\dot{\varphi}(t'') \rangle \\ &= \frac{d}{dt}\int_0^t dt' \int_0^t dt'' \langle F_\varphi(t') F_\varphi(t'') \rangle \\ &= \langle 2D_{\varphi\varphi} \rangle \ . \end{aligned} \tag{61}$$

Substituting Eq. (56) into Eq. (61) and using the expression (60) for the steady-state intensity, we find, after time integration,

$$\langle \varphi^2 \rangle = \frac{1}{4I_0} + \frac{\alpha+\gamma}{4I_0} t \ . \tag{62}$$

The integration constant $1/4I_0$ in Eq. (62) is determined by the contribution of vacuum fluctuations. These contributions are always present and have to be added to the noise produced by spontaneous emission.

Equation (62) is the famous Schawlow-Townes[8] result for a laser and states that the phase diffuses linearly in time. Note that this does not depend on the parameter p which we used to describe the fluctuations of the pump. Therefore the phase of the electromagnetic field is completely independent of the particular characteristic of the pump mechanism.

To determine the fluctuations in the quantity I we first linearize Eq. (55) around its steady state value. Defining $\Delta I = I - I_0$ and making again use of expression (60) for the steady-state intensity, we find

$$\frac{d}{dt}(\Delta I) = -\gamma\frac{\alpha-\gamma}{\alpha}\Delta I + F_I \ . \tag{63}$$

The Langevin Eq. (63) describes a very simple Markoff process, known as the Ornstein-Uhlenbeck process,[9,10] for which the steady-state variance is given by

$$\langle (\Delta I)^2 \rangle = D_{II} \Big/ \gamma\frac{\alpha-\gamma}{\alpha} \ . \tag{64}$$

Combining Eqs. (59), (64), and (57) we find our final expression for the photon number fluctuations

$$\langle (\Delta n)^2 \rangle = \left[\frac{\alpha}{\alpha-\gamma} - \frac{p}{4}\right] n_0 \ , \tag{65}$$

in which $n_0 = I_0$ is the average number of photons inside the cavity. As pointed out before the photon number fluctuations crucially depend on the particular pump mechanism. In the case of a Poisson statistic for the atomic injection times ($p=0$), the variance of the photon number is always larger than the mean number of photons. This is referred to as superpoissonian photon statistic. However, for a pump-noise suppressed laser ($p>0$), the variance can be smaller than n_0, which corresponds to photon number squeezed light. The optimum of such a noise reduction is achieved in the high-intensity limit by a regular injection of the atoms ($p=1$). In such a limit $\langle (\Delta n)^2 \rangle = 0.75 n_0$, which corresponds to 25% squeezing of the photon number noise.

It is interesting to notice that the amount of squeezing also depends on the relationship between the decay constants for the atomic levels $|a\rangle$ and $|b\rangle$. In the above analysis we assumed all the decay constants to be equal to that $\Gamma_a = \Gamma_b \equiv \Gamma$. In the more general case we have to allow for different values of the decay constants Γ_a, Γ_b, and Γ_{ab} for the atomic operators N_a, N_b, and M, respectively. The analysis for such a general case is completely analogous to the calculation above. The final result for the photon number fluctuation then acquires the form

$$\langle (\Delta n)^2 \rangle = \left[\frac{\alpha}{\alpha-\gamma} - \frac{\Gamma_b}{\Gamma_b+\Gamma_a}\frac{p}{2}\right] n_0 \ . \tag{66}$$

It is easy to see that Eq. (66) simplifies to Eq. (65) if $\Gamma_a = \Gamma_b$. However, if Γ_b is much larger than Γ_a, Eq. (66) becomes

$$\langle (\Delta n)^2 \rangle = \left[\frac{\alpha}{\alpha-\gamma} - \frac{p}{2}\right] n_0 \ . \tag{67}$$

If we again investigate the optimum case of a regular atomic injection ($p=1$), we see that it is now possible to achieve up to 50% squeezing of the photon number fluctuations. The limit $\Gamma_b \gg \Gamma_a$ is especially applicable to semiconductor lasers and our result is in complete agreement with the analysis by Yamamoto, Machida, and Nilson.[5]

V. SUMMARY

We have analyzed from first principle the effect of pump fluctuations in a laser through a Langevin operator

approach. We have generalized the usual quantum noise operators to incorporate the statistical properties of the pump mechanism. This allows us to discuss a large variety of pump fluctuations within a unified framework. We find that while the phase diffusion of the laser is unaffected by the pump noise, the intensity fluctuations crucially depend on the fluctuations of the pump mechanism. In the high-intensity limit it is possible to reduce the photon number fluctuations below the standard limit if the pump noise is smaller than that of a poisson distribution ($p>0$). Depending on the relationship between the atomic decay constants Γ_a and Γ_b, squeezing of the photon number fluctuations of up to 50% is possible.

Note added in proof. We are pleased to refer the reader to the excellent earlier paper on this problem by Golubev and Sokolov.[11] Unfortunately, this paper was not known to us and not referred to in our earlier publications.[3]

ACKNOWLEDGMENTS

This work has been partially supported by the Office of Naval Research (ONR). We also would like to thank Professor Y. Yamamoto for his helpful discussions and remarks. One of us (M.O.) would like to acknowledge the support of FONDECYT, Project No. 0363/88.

APPENDIX A

In this Appendix we want to show that

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S - R^2 = -pR\,\delta(t-t') \; . \tag{A1}$$

The subscript S on the bracket in Eq. (A1) indicates that we perform a statistical average over the injection times. The parameter p characterizes the injection statistics and fulfills the relation $p \leq 1$.

In order to evaluate the average in Eq. (A1) we have to specify the statistical properties of our injection model. For this we define a function $W(j,t_j;k,t_k)$ such that $W(j,t_j;k,t_k)dt_j dt_k$ is the joint probability that the jth atom enters the cavity during $[t_j,t_j+dt_j]$ and the kth atom enters during $[t_k,t_k+dt_k]$. Then the statistical average in Eq. (A1) can be written as

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S = \sum_{\substack{j,k \\ j\neq k}} \int_0^\infty dt_j \int_0^\infty dt_k\, W(j,t_j;k,t_k)\delta(t-t_j) \times \delta(t'-t_k) \; . \tag{A2}$$

Furthermore, we denote the marginal probability distribution of $W(j,t_j;k,t_k)$ by $W(j,t_j)$. Thus $W(j,t_j)dt_j$ is the probability that the jth atom enters the cavity during the time interval $[t_j,t_j+dt_j]$. Finally, let us label the atoms with positive, integer numbers such that the first atom is labeled 1, the second atom is labeled 2, and so on. To fix our time scale we assume that the first atom enters the cavity at $t=0$. Then the probability distribution for the first atom is given by

$$W(1,t_1)=\delta(t_1) \; . \tag{A3}$$

Furthermore, we note that

$$W(j,t_j)=0 \quad (t_j<0) \tag{A4}$$

because we started the atomic injection at $t=0$ by convention.

We next want to find a factorization rule of the joint probability distribution $W(j,t_j;k,t_k)$ into marginal distribution for individual atoms. For this we first write

$$W(j,t_j;k,t_k)=W(j,t_j|k,t_k)W(k,t_k) \; . \tag{A5}$$

Here $W(j,t_j|k,t_k)$ is the conditional probability, i.e., $W(j,t_j|k,t_k)dt_j$ gives the probability that the jth the atom arrives during $[t_j,t_j+dt_j]$, provided the kth atom arrived at t_k.

If our injection process is stationary and has reached a steady-state configuration the conditional probability should only depend on the differences $j-k$ and t_j-t_k. We can therefore write

$$W(j,t_j|k,t_k)=W(j-k+1,t_j-t_k|1,0) \quad (\text{for } j>k) \; . \tag{A6}$$

Here and in the following discussion we adopt the convention that j is larger than k. We next note that the conditional probability $W(l,t_l|1,0)$ is equal to the marginal distribution $W(l,t_l)$. This is due to the fact that we assumed the first atom to enter the cavity at $t=0$. We can now use this identity and obtain from Eqs. (A6) and (A5)

$$W(j,t_j;k,t_k)=W(j-k+1,t_j-t_k)W(k,t_k) \quad (\text{for } j>k) \; . \tag{A7}$$

The expression Eq. (A7) is the desired factorization of the joint probability distribution into a product of marginal distributions. We can now use this expression to simplify Eq. (A2). We first write

$$\sum_{\substack{j,k \\ j\neq k}} \int_0^\infty dt_j \int_0^\infty dt_k\, W(j,t_j;k,t_k)\delta(t-t_j)\delta(t'-t_k) = \sum_{\substack{j,k \\ j>k}} \int_0^\infty dt_j \int_0^\infty dt_k\, W(j,t_j;k,t_k)\delta(t-t_j)\delta(t'-t_k) + \sum_{\substack{j,k \\ j>k}} \int_0^\infty dt_j \int_0^\infty dt_k\, W(k,t_k;j,t_j)\delta(t-t_k)\delta(t'-t_j) \; . \tag{A8}$$

We next use the fact that $W(j,t_j;k,t_k)=W(k,t_k;j,t_j)$ and obtain with the help of Eq. (A7)

$$\sum_{j>k}\int_0^\infty dt_j\int_0^\infty dt_k\,W(j,t_j;k,t_k)[\delta(t-t_j)\delta(t'-t_k)+\delta(t'-t_j)\delta(t-t_k)]$$
$$=\sum_{j>k}\int_0^\infty dt_j\int_0^\infty dt_k\,W(j-k+1,t_j-t_k)W(k,t_k)[\delta(t-t_j)\delta(t'-t_k)+\delta(t'-t_j)\delta(t-t_k)]\ . \quad \text{(A9)}$$

Recall that $W(j,t_j)=0$ for $t_j<0$. Now we see that $W(j-k+1,t_j-t_k)$ is zero unless $t_j-t_k>0$, which is consistent with the requirement that $j>k$. Therefore the integral over t_j effectively begins at $t_j=t_k$. If we denote t_j-t_k by s we find

$$\sum_{j>k}\int_0^\infty dt_k\int_0^\infty dt_j\,W(j-k+1,t_j-t_k)W(k,t_k)[\delta(t-t_j)\delta(t'-t_k)+\delta(t'-t_j)\delta(t-t_k)]$$
$$=\sum_{j>k}\int_0^\infty dt_k\int_0^\infty ds\;W(j-k+1,s)W(k,t_k)[\delta(t'-t_k)\delta(t-t_k-s)+\delta(t-t_k)\delta(t'-t_k-s)]\ . \quad \text{(A10)}$$

We note that we are only interested in times t and t' larger than zero. The integration over the δ functions can then be easily performed and we obtain from Eq. (A10)

$$\sum_{j>k}\int_0^\infty ds\;W(j-k+1,s)[W(k,t')\delta(t-t'-s)+W(k,t)\delta(t'-t-s)]$$
$$=\sum_{j>k}\begin{cases}W(j-k+1,t-t')W(k,t') & \text{for } t-t'>0\\ W(j-k+1,t'-t)W(k,t) & \text{for } t-t'<0\end{cases}$$
$$=\sum_{j>k}W(j-k+1,|t-t'|)W(k,t_<)\ , \quad \text{(A11)}$$

with

$$t_<=\min(t,t')\ . \quad \text{(A12)}$$

The expression in Eq. (A12) can be simplified even further

$$\sum_{j>k}W(k,t_<)W(j-k+1,|t-t'|)$$
$$=\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty}W(k,t_<)W(j-k+1,|t-t'|)$$
$$=\sum_{k=1}^{\infty}\sum_{l=2}^{\infty}W(k,t_<)W(l,|t-t'|)$$
$$=\left[\sum_{k=2}^{\infty}W(k,t_<)+W(1,t_<)\right]\left[\sum_{k=2}^{\infty}W(l,|t-t'|)\right]\ . \quad \text{(A13)}$$

We next use Eq. (A3) and the definition of $t_<$ and remark that $W(1,t_<)$ is always zero. This is due to the fact that we assumed t and t' to be larger than zero. We therefore find the simple expression

$$\left\langle\sum_{\substack{j,k\\ j\neq k}}\delta(t-t_j)\delta(t'-t_k)\right\rangle_S=f(t_<)f(|t-t'|)\ , \quad \text{(A14)}$$

with

$$f(s)=\sum_{l=2}^{\infty}W(l,s)\ . \quad \text{(A15)}$$

Physically, the quantity $f(s)ds$ is the probability that any atom, apart from the first one, is injected into the cavity during the time interval $[s,s+ds]$. In Appendix B we have related the quantity $f(s)$ to more familiar statistical quantities. We now evaluate and discuss $f(s)$ for several cases of particular interest.

1. Regular injection

In this case the time between consecutive atoms is fixed so that there are no fluctuations in the number of atoms which are injected during a fixed time interval. We note that $1/R$ is the time between consecutive atoms in which R is the mean atomic injection rate. Therefore the jth atom is injected at time $(j-1)/R$ is we again assume that the first atom is injected at $t=0$. The marginal probability distribution $W(j,s)$ is then given by

$$W(j,s)=\delta\left[s-\frac{j-1}{R}\right]\ . \quad \text{(A16)}$$

Substituting Eq. (A16) into the definition of $f(s)$ we find

$$f(s)=\sum_{l=2}^{\infty}\delta\left[s-\frac{l-1}{R}\right]=\sum_{l=1}^{\infty}\delta\left[s-\frac{l}{R}\right]$$
$$=\sum_{l=0}^{\infty}\delta\left[s-\frac{l}{R}\right]-\delta(s)\ . \quad \text{(A17)}$$

The time scale we are interested in is much larger than the time between two consecutive atoms. Therefore we can change the summation in Eq. (A17) into an integration and obtain

$$\sum_{l=0}^{\infty}\delta\left[s-\frac{l}{R}\right]=R\int_0^\infty d\left[\frac{1}{R}\right]\delta\left[s-\frac{1}{R}\right]=R\ . \quad \text{(A18)}$$

Substituting Eq. (A18) into (A17), we obtain

$$f(s)=R-\delta(s)\ . \quad \text{(A19)}$$

Therefore Eq. (A14) simplifies to

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S = [R-\delta(t_<)][R-\delta(|t-t'|)]$$

$$= R[R-\delta(t-t')] \ . \qquad \text{(A20)}$$

In the last step we again used the fact that $t_<$ is larger than zero and therefore the first δ function vanishes. Subtracting the term R^2 on both sides of Eq. (A20) yields our final expression for the case of a regular injection

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S - R^2 = -R\delta(t-t') \ . \qquad \text{(A21)}$$

This is just Eq. (A1) in which the parameter p has the value 1.

2. Poissonian injection

In this case the number of atoms, which have been injected into the cavity at a given time, follows a Poisson statistic. The marginal probability distribution $W(j,s)$ is then given by[9,10]

$$W(j,s) = RP(j-1,s) \quad (\text{for } j>1) \ . \qquad \text{(A22)}$$

Here R is again the average injection rate of the atoms. The function $P(l,t)$ is the probability that at time t, l atoms have entered the cavity. Thus $P(l,t)$ is the Poisson distribution

$$P(l,t) = \frac{(Rt)^{l-1}}{(l-1)!} e^{-Rt} \ . \qquad \text{(A23)}$$

The factor $l-1$ stems from our assumption that at $t=0$ already one atom has entered the cavity. Substituting Eqs. (A22) and (A23) into the definition for the function $f(s)$ yields

$$f(s) = \sum_{l=2}^{\infty} R \frac{(Rs)^{l-2}}{(l-2)!} e^{-Rs} = R \ . \qquad \text{(A24)}$$

Hence we see that $f(s)$ is independent of time. The final expression in the case of a Poissonian injection is then given by

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S - R^2 = 0 \ , \qquad \text{(A25)}$$

i.e., $p=0$ in Eq. (A1).

3. General case

As mentioned before, $f(s)ds$ is the probability that any atom, which is different from the first one, is injected into the cavity during $[s, s+ds]$. It is intuitively clear that for times $t \gg 0$ this probability should be time independent, at least on a coarse-grained time scale (see also Appendix B). This is necessary to guarantee a steady state for the average number of injected atoms during a fixed time interval. Therefore we expect that for large times $f(s)$ approaches the constant, average injection rate R. However, on a time scale of order $1/R$, which is the average time between consecutive atoms, $f(s)$ specifies the amount of atomic bunching. In the case of a regular injection we have complete antibunching so that $f(s)$ goes to zero as s approaches zero. On the other hand, a Poisson distribution has a strong bunching effect so that the injection probability $f(s)$ remains constant when s approaches zero [c.f. Eq. (A24)]. In the general case it is physically reasonable to make the ansatz

$$f(s) = R(1-pe^{-Rs}) \quad (s \geq 0) \ . \qquad \text{(A26)}$$

The parameter p quantifies the amount of antibunching of the atoms. For $p=0$ we have the Poissonian case (A24). With increasing values of p the atoms become more and more antibunched, which reaches its maximum in the case of a regular injection ($p=1$). It is also clear from Eq. (A26) that p cannot be larger than 1 because $f(s)$ has to be non-negative.

Substituting Eq. (A26) into Eq. (A14) yields

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S = R^2(1-pe^{-Rt_<})(1-pe^{-R|t-t'|})$$

$$= R^2(1-pe^{-R|t-t'|}) \ . \qquad \text{(A27)}$$

In the last step we have made use of our assumption that the time t and t' are much larger than zero (in fact $t,t' \gg 1/R$). This allows us to neglect transient effects which arise from the initial condition of our pumping mechanism. Furthermore, if we are interested in a time scale that is much larger than $1/R$, we may approximate $Re^{-R|t-t'|}$ by the δ function $\delta(t-t')$. We then obtain from Eq. (A27)

$$\left\langle \sum_{\substack{j,k \\ j\neq k}} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S - R^2 = -pR\delta(t-t') \ . \qquad \text{(A28)}$$

This is just the relation (A1) we set out to prove.

APPENDIX B

In this Appendix we want to establish a relationship between the quantity $f(s)$ in Appendix A and more familiar statistical quantities for a given injection mechanism. Let again $P(l,t)$ be the probability that at time t, l atoms have entered the cavity. Then it is easy to see that $P(l,t)$ and the marginal distribution $W(l,t)$ are related by

$$\frac{d}{dt}P(l,t) = W(l,t) - W(l+1,t) \ . \qquad \text{(B1)}$$

Equation (B1) simply states that the change of probability is equal to the "rate in" minus the "rate out." Solving this expression for $W(l+1,t)$ and iterating the equation yield

$$W(l+1,t) = -\frac{d}{dt}\sum_{j=0}^{l} P(j,t) = \frac{d}{dt}\sum_{j=l+1}^{\infty} P(j,t) \ . \qquad \text{(B2)}$$

In the last step we made use of the fact that the sum over all probabilities is equal to 1. We then find for $f(s)$

$$f(s)=\sum_{l=2}^{\infty} W(l,s)=\frac{d}{ds}\sum_{l=2}^{\infty}\sum_{j=l}^{\infty} P(j,s)$$
$$=\frac{d}{ds}\sum_{j=2}^{\infty}\sum_{l=2}^{j} P(j,s)$$
$$=\frac{d}{ds}\sum_{j=0}^{\infty} jP(j+1,s)\ . \quad \text{(B3)}$$

The quantity $\sum_{j=0}^{\infty} jP(j+1,s)$ is the average number of atoms, different from atom 1, which have entered the cavity at time s. Our function $f(s)$ is just the time derivative of this statistical quantity. Note also that in our case

$$\frac{d}{ds}\sum_{j=0}^{\infty} jP(j+1,s)=\frac{d}{ds}\sum_{j=0}^{\infty} jP(j,s) \quad (\text{for } s>0)\ , \quad \text{(B4)}$$

so that the function $f(s)$ can be easily computed from the first moment of the probability distribution. This interpretation of $f(s)$ supports our discussion of the general case in Appendix A.

---

*Permanent address: Central Research Institute for Physics, H-1525 Budapest 114, P.O. Box 49, Hungary.

†Permanent address: Departamento de Fisica, Pontificia Universidade Catolica, Caixa Postal 38071, 22453, Rio de Janeiro, Brazil.

‡Permanent address: Hunter College of the City University, New York, NY 10021.

§Permanent address: Facultad de Fisica Pontificia, Universidad Catolica de Chile, Casilla 114-D, Santiago, Chile.

[1]For experiments on squeezing, see, for example, R. E. Slusher, L. W. Hollbertg, B. Yurke, J. C. Mertz, and J. F. Walley, Phys. Rev. Lett. **55**, 2405 (1985); R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. DeVoe, and D. F. Walls, *ibid.* **57**, 691 (1986); L. A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, *ibid.* **57**, 2520 (1986). For the theoretical aspects of squeezing, see D. F. Walls, Nature **306**, 141 (1983); B. L. Schumaker and C. M. Caves, Phys. Rev. A **31**, 3093 (1985); Y. Yamamoto, S. Machida, and O. Nilsson, *ibid.* **34**, 4025 (1986); W. Schleich and J. A. Wheeler, Nature (London) **356**, 574 (1987).

[2]M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985); M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987); J. Bergou, M. Orszag, and M. O. Scully, *ibid.* **38**, 754 (1988); J. Bergou and M. Orszag, *ibid.* **38**, 763 (1988); J. Bergou, M. Orszag, and M. O. Scully, *ibid.* **38**, 768 (1988). For comparison to experiment, see also J. Bergou and M. Orszag, J. Opt. Soc. Am. B **5**, 249 (1988).

[3]J. Bergou, L. Davidovich, M. Orszag, C. Benkert, M. Hillery, and M. O. Scully, Phys. Rev. A **40**, 5073 (1989).

[4]For a different approach to the pump noise, see Y. Yamamoto, S. Machida, and O. Nilsson, Phys. Rev. A **34**, 4025 (1986); S. Machida, Y. Yamamoto, and Y. Itaya, Phys. Rev. Lett. **58**, 1000 (1987); Y. Yamamoto and S. Machida, Phys. Rev. A **35**, 5114 (1987); M. A. Marte, H. Ritsch, and D. F. Walls, Phys. Rev. Lett. **61**, 1093 (1988).

[5]Y. Yamamoto, S. Machida, and O. Nilson, Phys. Rev. A **34**, 4025 (1986).

[6]See, for example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison Wesley, Reading, MA, 1974).

[7]See, for example, M. Lax, in *Statistical Physics, Phase Transition and Superconductivity,* edited by M. Chrétien, E. P. Gross, and S. Dreser (Gordon and Breach, New York, 1968), Vol. II, p. 425.

[8]A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958); J. P. Gordon, H. J. Zeiger, and C. H. Townes, *ibid.* **99**, 1264 (1955).

[9]N. G. Van Kampen, in *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[10]C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1983).

[11]M. Golubev and I. V. Sokolov, Zh. Eksp. Teor. Fiz. **87**, 408 (1984) [Sov. Phys.—JETP **60**, 234 (1984)].