Reconstruction of the state of the radiation field in a cavity through measurements of the outgoing field

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We establish the relation between the Wigner function of a field emerging from a cavity, determined through pulsed homodyne tomography, and the phase-space distribution of the original field in the cavity. We show that the reconstructed Wigner function of the outgoing field is equivalent to generalized phase-space distributions of the initial cavity field, the original Wigner distribution being obtained when there is perfect mode matching between the outgoing field and the homodyning pulse.

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I. INTRODUCTION

The characterization of radiation has evolved in recent years from the measurement of variances, spectra, and photon-number populations to the determination of the full quantum state of the field, as described either by the density matrix in the photon-number basis, or by phase-space distributions. These distributions allow the calculation of quantum averages of functions of field operators in a classical-like way, as if the operators were *c*-numbers. Different distributions are obtained, depending on the operator ordering adopted for defining the quantum-classical correspondence [1]. Of special interest is the Wigner distribution [2], which, as shown by Moyal [3], corresponds to a symmetrical ordering of the field annihilation and creation operators. Indeed, the proposal of the method of optical homodyne measurement by Yuen and Chan [4] and Abbas *et al.* [5] led to the measurement of the probability distribution of field quadratures of propagating radiation [6], from which it is possible, as shown by Vogel and Risken [7,8], to reconstruct the Wigner function of the field. The homodyning technique corresponds to combining, in a beam splitter, the field to be measured with an essentially classical field (''local oscillator''), which could be eventually a pulse [6,9], and detecting the difference in photon counting between two detectors placed at the two outgoing ports of the beam splitter. This difference is proportional to a generalized quadrature of the field along a direction in phase space determined by the relative phase between the local oscillator and the field to be measured. The reconstruction is made by applying, to the probability distributions of generalized quadratures of the field for several directions in phase space, an integral transform introduced by Radon [10], which is the basis of medical tomography. This quantum reconstruction technique, known as quantum tomography, has been successfully demonstrated in recent experiments [6,11], which resulted in the mapping of the Wigner functions of coherent states, squeezed states, as well as incoherent superpositions of coherent states, such as phase-diffused, amplitude-diffused, and chaotic light [12]. The Wigner distribution can be thought as a joint distribution for the field quadratures: when integrated with respect to one of the quadratures x_θ, it yields the probability distribution of the orthogonal quadrature $x_{\theta+\pi/2}$. However, it cannot be interpreted as a true probability distribution in phase space: even though it is real and bounded, it may assume negative values. This is not the case, however, for the distributions measured so far by the technique of quantum tomography: the Wigner functions of coherent and squeezed states are positive-definite, and therefore these fields can be understood as classical fluctuating quantities.

On the other hand, recent experiments in cavity quantum electrodynamics have led to the realization of states of fields in cavities which correspond to negative-valued Wigner functions, like coherent superpositions of two coherent states (''Schrödinger-cat-like states'') [13,15] and Fock states [14]. No interpretation in terms of classical fluctuating fields is possible in this case. The measurement of the Wigner function of these states would then be a quite stringent test of the quantum nature of an electromagnetic field. Procedures for measuring the Wigner function of a field in a cavity by detecting the internal state of atoms which cross the cavity and interact with the field have been presented by many authors [16–18]. As opposed to the optical homodyne method, appropriate for running waves, these proposals do not require the calculation of integral transforms, and yield in a more direct way the quantum state of the field. Recently, following the proposal in Ref. [17], the Wigner function for a single-photon field was measured at the origin of phase space [19]. Its value is negative, thus exhibiting the nonclassical nature of this state.

Measurements of quantum states of cavity fields must be made in a time short compared with the decoherence time, which in the case of Schrödinger-cat-like states is of the order of the cavity dissipation time divided by the distance between the two states in phase space (roughly of the order of the average number of photons in the state). As the field leaks into the environment, the quantum characteristics of the state are washed out. While it is still possible to reconstruct the original state of the field in the cavity after it has started to decay [20], the requirements on the precision of the measurement increase as dissipation has more time to act on the state. On the other hand, since coherence is leaked into the environment, it should be possible to reconstruct the state of the field in the cavity by measurements made on the outgoing field. In order to understand this process, it is useful to

consider a simple model in which the losses correspond to the leaking of the intracavity field through a partially transmitting mirror. A special case, in which the environment was replaced by a single harmonic oscillator (another cavity, coupled to the first one through a waveguide), was considered in Ref. [21]. In this paper we consider a continuum of field modes coupled to the cavity mode, and discuss the determination of the Wigner function of the field inside the cavity through homodyne measurements made on the outgoing field. Of course, in a real experiment, the dissipation of the intracavity field cannot be attributed solely to the transmission of the field to the space outside the cavity. Furthermore, diffraction losses, by far the most important loss mechanism, result in a wide distribution of very few outgoing photons, which are therefore very difficult to detect, and the process cannot be mimicked by a partially transmitting mirror. A simple model helps, however, to answer some questions of principle, motivated by the two different possibilities of measuring the quantum state of a field: either through homodyning, for running fields, or through atomic measurements, as proposed for cavity fields. How precisely can we determine the initial state inside the cavity by measuring the Wigner function of the traveling pulse? How does this distribution relate to the one corresponding to the field inside the cavity? What is the best choice for the homodyning field? What happens when another choice is made? Can the complete quantum state of the field inside the cavity always be recovered, independently of the form of the homodyning pulse?

In order to answer these questions, we relate in this paper the Wigner function of the outgoing field, obtained through quantum tomography via pulsed homodyne detection, with the Wigner distribution of the intracavity field. The spirit of this work is the same as the one of the seminal paper by Collet and Gardiner [22], which related the time and normal-ordered correlation functions of the outgoing field with the analogous functions for the intracavity field, thus allowing one to express the measured outside spectrum in terms of correlation functions of cavity-mode operators. The model adopted here is actually closely related to the one in that reference. In this work, however, we establish the relation between the full quantum state of the field inside the cavity and the information gathered on the outside field through the homodyning technique.

In the model here considered, the radiation field is described by two kinds of field operators, corresponding respectively to the intracavity mode under consideration and the continuum of modes outside the cavity. The cavity mode is coupled to the external modes through a phenomenological linear interaction. The model is described in detail in Sec. II. The evolution of the field inside and outside the cavity is determined in Sec. III. In Sec. IV we show that a homodyne detection of the outgoing field yields generalized distributions for the initial field inside the cavity. Furthermore, for a proper choice of the homodyning field, the measured distribution may coincide exactly with the initial state of the field in the cavity. Two examples are discussed in Sec. V, and our conclusions are summarized in Sec. VI.

II. THE MODEL

We consider a one-dimensional optical cavity of size L, located in the region $[-L,0]$ and such that, for fields in a certain range of frequencies one of its mirrors, located at $x=-L$, is ideal (perfect) with reflectivity $r_1=1$, while the other, located at $x=0$, is almost perfect, having a reflectivity $r_2 \simeq 1$. In the limit where both mirrors are perfect, $r_1=r_2=1$, the normal modes of the cavity and of the external world are independent.

For a perfect cavity ($r_1=r_2=1$) we may define a set of discrete numerable modes inside the cavity and an independent set of continuous outside modes that are independent. We consider, for simplicity, a one-dimensional model, with fields characterized by a single wave number and single polarization, and propagating outside the cavity along a single direction. As a matter of fact, for the internal modes, the wave number just characterizes the mode under consideration, which does not have to be one dimensional. On the other hand, one should note that, for the external modes, the one-dimensional model can actually be justified in terms of the paraxial approximation, so long as the beam propagates primarily in one direction and has a bandwidth much less than its central frequency [9]: the transverse contributions are integrated over the detector surface.

The internal modes are associated with discrete field annihilation and creation operators $\{a_j\}$ and $\{a_k^\dagger\}$ satisfying the commutation relation $[a_j,a_k^\dagger]=\delta_{jk}$, while the annihilation and creation operators associated with the external modes form a continuous set labeled by the corresponding frequencies Ω and obeying the commutation relations $[b(\Omega),b^\dagger(\Omega')]=\delta(\Omega-\Omega')$. The independence of the field modes implies that $[a_j,b(\Omega)]=[a_k,b^\dagger(\Omega)]=0$. The field operators inside and outside the cavity should obey the proper boundary conditions and in the limit of a perfect cavity they are given, in the Schrödinger picture, by

$$E_{\mathrm{cav}}(x)=\sum_j \sqrt{\frac{\hbar\omega_j}{L\epsilon_0}}(a_j+a_j^\dagger)\sin(\omega_j x/c), \tag{1}$$

$$E_{\mathrm{ext}}(x)=\int \sqrt{\frac{\hbar\Omega}{\pi\epsilon_0 c}}[b(\Omega)+b^\dagger(\Omega)]\sin(\Omega x/c)d\Omega, \tag{2}$$

where $\omega_j=j\pi c/L$, $j=1\ldots$, and c is the velocity of light.

We assume that the cavity finesse is high enough, although finite (so that the mode width is much smaller than the mode separation), and that the initial field is in one of the internal modes, with frequency ω_{cav}. The finite transmissivity of the mirror couples the modes inside and outside the cavity, allowing the creation of a photon outside through the annihilation of one photon inside the cavity and *vice versa*. The weakness of the coupling implies that the modes of the internal and external fields are still represented approximately by Eqs. (1) and (2). The corresponding Hamiltonian is given by

$$H=\hbar\omega_{\rm cav}a^{\dagger}a+\int\hbar\Omega b^{\dagger}(\Omega)b(\Omega)d\Omega + \int\hbar G(\Omega)[a^{\dagger}b(\Omega)+ab^{\dagger}(\Omega)]d\Omega, \quad (3)$$

where the first two terms represent the free-field Hamiltonians for the internal and external fields, and the last one describes the interaction mediated by the mirror. The form factor $G(\Omega)$, taken to be real for simplicity, may be thought of as representing the frequency-dependent mirror transmission function.

As a matter of fact, when the mirror is not perfect, the two regions cannot be considered as independent anymore, and the field modes should be defined for the whole space, taking into consideration the boundary conditions associated with the cavity mirror [23]. However, for high-Q cavities, such that the mode linewidth is much smaller than the intermode distance, it is a good approximation to define interacting internal and external modes, and to neglect the change in the spatial dependence of these modes.

We proceed to calculate now, from Eq. (3), the time evolution of the field operators.

III. EVOLUTION OF THE FIELD OPERATORS

In the Heisenberg picture, the time evolution of the field operators $a(t)$ and $b(\Omega,t)$ is described by the differential equations

$$\frac{da(t)}{dt}=-i\omega_{\rm cav}a(t)-i\int G(\Omega)b(\Omega,t)d\Omega, \quad (4)$$

$$\frac{db(\Omega,t)}{dt}=-i\Omega b(\Omega,t)-iG(\Omega)a(t), \quad (5)$$

with the initial conditions $a(0)=a_0$ and $b(\Omega,0)=b_0(\Omega)$. Taking the Laplace transform of these equations we get

$$s\tilde{a}(s)-a_0=-i\omega_{\rm cav}\tilde{a}(s)-i\int d\Omega G(\Omega)\tilde{b}(\Omega,s), \quad (6)$$

$$s\tilde{b}(\Omega,s)-b_0(\Omega)=-i\Omega\tilde{b}(\Omega,s)-iG(\Omega)\tilde{a}(s), \quad (7)$$

where $\tilde{a}(s)$ and $\tilde{b}(\Omega,s)$ are the Laplace transforms of $a(t)$ and $b(\Omega,t)$. From Eqs. (6) and (7) we have

$$\tilde{a}(s)=\frac{a_0}{s+\Gamma/2+i(\omega_{\rm cav}+\delta)}-\frac{i}{s+\Gamma/2+i(\omega_{\rm cav}+\delta)} \times\int d\Omega G(\Omega)\frac{b_0(\Omega)}{s+i\Omega}, \quad (8)$$

and

$$\tilde{b}(\Omega,s)=\frac{b_0(\Omega)}{s+i\Omega}-i\frac{G(\Omega)}{s+i\Omega}\tilde{a}(s), \quad (9)$$

where

$$\Gamma(s)/2+i\delta(s)=\int_0^{\infty}d\Omega\frac{G^2(\Omega)}{s+i\Omega}.$$

We assume that $G(\Omega)$ is a broad function of Ω, centered around $\omega_{\rm cav}$. In this case $\delta(s)$ and $\Gamma(s)$ should be approximately independent of s. For example, taking $G(\Omega)$ as a Lorentzian,

$$G(\Omega)=g\frac{\lambda^2}{(\Omega-\omega_{\rm cav})^2+\lambda^2},$$

and extending the integration from $-\infty$ to ∞, we have

$$\Gamma(s)/2+i\delta(s)=-\frac{\pi\lambda g^2}{2}\frac{2\lambda-i(\omega_{\rm cav}-is)}{[\omega_{\rm cav}-i(\lambda+s)]^2}. \quad (10)$$

Assuming that Γ and δ are constants and small compared to λ and $\omega_{\rm cav}$, respectively, the largest contributions for $\tilde{a}(s)$ come from a pole close to $s=i(\omega_{\rm cav})-\Gamma/2$, resulting, consistently, in $\Gamma\approx 2\pi g^2$ and $\delta\ll\omega_{\rm cav}$. These conditions correspond to the Markov approximation. The quantity δ is a small frequency shift that renormalizes the frequency $\omega_{\rm cav}$ to $\omega_0=\omega_{\rm cav}+\delta$ and Γ corresponds to the decay constant of the field inside the cavity, as we will show below. Taking the inverse Laplace transform of $\tilde{a}(s)$ we obtain

$$a(t)=f(t)a_0+\int g(\Omega,t)b_0(\Omega)d\Omega, \quad (11)$$

where

$$f(t)=e^{-i\omega_0 t-(\Gamma/2)t}, \quad (12)$$

and

$$g(\Omega,t)=G(\Omega)\frac{(e^{-i\omega_0 t-\Gamma/2t}-e^{-i\Omega t})}{\omega_0-\Omega-i\Gamma/2}. \quad (13)$$

For the inverse Laplace transform of $\tilde{b}(\Omega,s)$ we obtain

$$b(\Omega,t)=g(\Omega,t)a_0+\int h(\Omega,\Omega',t)b_0(\Omega')d\Omega', \quad (14)$$

where

$$h(\Omega,\Omega',t)=e^{-i\Omega' t}\delta(\Omega-\Omega')+G(\Omega)G(\Omega') \times\left\{\frac{1}{(\Omega'-\Omega)}\left[\frac{e^{-i\Omega t}}{(\omega_0-\Omega-i\Gamma/2)} -\frac{e^{-i\Omega' t}}{(\omega_0-\Omega'-i\Gamma/2)}\right] +\frac{e^{-i\omega_0 t-i\Gamma/2t}}{(\Omega-\omega_0+i\Gamma/2)(\Omega'-\omega_0+i\Gamma/2)}\right\}. \quad (15)$$

An immediate application for these time-evolved operators is the calculation of time-dependent normally ordered correlation functions for both internal and external operators. We recover then the relation established in [22]. In order to compare our results with those of Ref. [22], we define an outgoing operator $b_{out}(t)$, given by

$$b_{out}(t)=i\sqrt{\frac{2}{\pi}}\int b(\Omega,t)d\Omega. \tag{16}$$

Assuming that the initial state is a product of an arbitrary state inside the cavity and the vacuum outside it, one gets from Eqs. (11), (14), and (16),

$$\begin{aligned}\langle b_{out}^{\dagger}(t_1)\cdots b_{out}^{\dagger}(t_n)b_{out}(t_{n+1})\cdots b_{out}(t_{n+m})\rangle \\ =\Gamma^{(m+n)/2}\langle\tilde{T}[a^{\dagger}(t_1)\cdots a^{\dagger}(t_n)] \\ \times T[a(t_{n+1})\cdots a(t_{n+m})]\rangle,\end{aligned} \tag{17}$$

where $T,\tilde{T}$ are time-ordering and antiordering operators, respectively. This relation is exactly the one found in [22], as it should be since the model so far is very similar to the one developed there. However, as we have already stressed, we will be interested in calculating more than just normally ordered correlation functions. Our goal is to relate the Wigner function of the inside field with measurements done on the outside field.

More generally, Eqs. (11) and (14) yield the dynamics of any field observable of the internal and external fields, relating its value at $t>0$ to the corresponding initial value ($t=0$). In particular, they can be used to calculate, at any time t, the characteristic functions for the internal and external fields [24].

The normal-ordered characteristic function for the internal field is given by

$$C_N^{\rm cav}(\lambda,\lambda^*)={\rm tr}\{\rho e^{\lambda a^{\dagger}}e^{-\lambda^* a}\}, \tag{18}$$

where ρ is the density operator of the field. The function C_N contains all information about the field inside the cavity and provides mean values of cavity operators in the normal order by simple derivatives over λ and λ^*.

The Wigner function for the internal field is obtained by Fourier transforming the symmetric characteristic function $C^{\rm cav}(\lambda,\lambda^*)=C_N^{\rm cav}(\lambda,\lambda^*)e^{-|\lambda|^2/2}$ [1],

$$W^{\rm cav}(\eta,\eta^*)=\int\frac{d^2\lambda}{\pi}e^{\eta^*\lambda-\eta\lambda^*}C^{\rm cav}(\lambda,\lambda^*). \tag{19}$$

If the initial field is totally concentrated inside the cavity, that is the total density operator is initially given by $\rho=\rho_{in}\otimes\rho_{ext}$, with $\rho_{ext}=|0\rangle\langle 0|$ being the vacuum state for all external modes, we obtain, using Eq. (11) in Eq. (18),

$$\begin{aligned}C_N^{\rm cav}(\lambda,\lambda^*,t)&={\rm tr}[\rho e^{\lambda f^*(t)a_0^{\dagger}}e^{-\lambda^* f(t)a_0}] \\ &=C_N^{\rm cav}[\lambda f^*(t),\lambda^* f(t),0].\end{aligned} \tag{20}$$

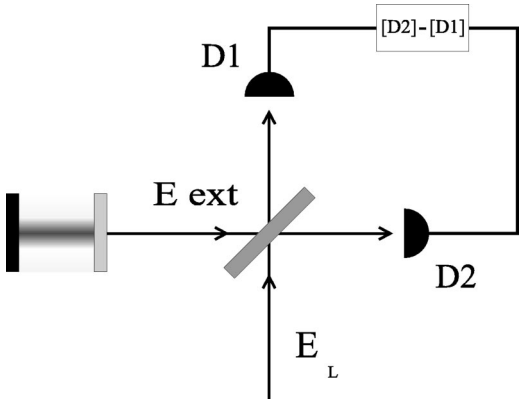

FIG. 1. Homodyne scheme. The intracavity field leaks through the semitransparent mirror and is mixed with the local oscillator field at the beam splitter. The signals detected at $D1$ and $D2$ are subtracted.

This equation shows that the time evolution of the characteristic function inside the cavity is equivalent to a change of scale on its parameters. Therefore, the time evolution of any normal-ordered product is given by

$$\begin{aligned}\langle a^{\dagger m}a^n\rangle_t&=f^*(t)^m f(t)^n\langle a^{\dagger m}a^n\rangle_0 \\ &=e^{i\omega_0(m-n)t}e^{-\Gamma(m+n)t/2}\langle a^{\dagger m}a^n\rangle_0.\end{aligned} \tag{21}$$

For the external modes, a more careful procedure is needed, since they are defined over a continuum. We turn therefore to an operationally defined phase-space distribution, obtained through tomographic methods from homodyne measurements of the outgoing field.

IV. THE WIGNER FUNCTION FOR THE FIELD OUTSIDE THE CAVITY

We derive in this section the phase-space distribution obtained from homodyne measurements on the external field. In doing so, one should take into account that any measurement deals not with a single frequency but with some frequency band, defined both by the detectors and the field hitting them. We assume that the field which leaves the cavity, resulting from the decay of the internal field, is homodyned with an intense, classical pulse. The two fields are combined in a beam splitter as shown in Fig. 1 and the two resulting signals coming from the beam splitter, and detected by detectors D_1 and D_2 are subtracted. The result is proportional to a quadrature of the quantum field. We consider here balanced homodyning (50% transmission of the beam splitter), photon flux-sensitive detectors that do not distinguish the photon energy, and a dc detection where the signals on the detector are integrated for a time long enough for the entire pulse to be detected [9].

The external field operator in the Heisenberg picture is given by

$$E_{\rm ext}(x,t)=\int\sqrt{\frac{\hbar\Omega}{\pi\epsilon_0 c}}[b(\Omega,t)+b^{\dagger}(\Omega,t)]\sin(\Omega x/c)d\Omega,$$

while the pulse field of the local oscillator is taken as

$$E_L(x,t)=E_L^+(x,t)+E_L^-(x,t),$$

where $E_L^+(x,t)$ is the positive-frequency part of the local field,

$$E_L^+(x,t)=\int \mathcal{E}_\varphi(\omega)e^{-i\omega(t-x/c)}d\omega, \tag{22}$$

and $E_L^-(x,t)=E_L^+(x,t)^*$. $\mathcal{E}_\varphi(\omega)$ is analytic in the upper half plane so that $E_L^+(x,t)$ is zero for $t-x/c<0$. We also assume that $\mathcal{E}_\varphi(\omega)$ has a peak, with width γ, at a frequency ω_L close to $\omega_{\rm cav}$, when its phase is φ. For example, we may take it as having a Lorentzian spectrum

$$\mathcal{E}(\omega)=\frac{1}{2\pi}\frac{iE_0e^{-i\varphi}}{(\omega-\omega_L)+i\gamma/2}. \tag{23}$$

In this case we have

$$E_L^+(x,\tau)=E_0e^{-i\omega_L(t-x/c)-(\gamma/2)t-i\varphi}\theta(t-x/c).$$

Let l be the path length of both pulses to the detectors. The signal measured in each detector is proportional to an integral of the total normal-ordered intensity $I(l,t)=:|E_L(l,t)+E_{\rm ext}(l,t)|^2$: over T, the time window of the detector, assumed to be much larger than $1/\Gamma$ and $1/\gamma$. The difference in the integrated intensities in detectors D_1 and D_2 is proportional to an average quadrature $B_\varphi+B_\varphi^\dagger$, defined by

$$B_\varphi=Z\int_0^T E_L^-(0,\tau)d\tau\int d\Omega\sqrt{\Omega}b(\Omega,\tau)e^{i\Omega l/c}, \tag{24}$$

where Z is chosen as a real and positive normalization constant such that $[B_\varphi,B_\varphi^\dagger]=1$. Notice that in Eq. (24) there appears $e^{i\Omega l/c}$, which corresponds to the part of E_{ext} that contributes to the detected fluxes. Substituting the value of $b(\Omega,t)$ given by Eq. (14) into Eq. (24) we have

$$B_\varphi=F_{\rm cav}^*a_0+\int F_{\rm ext}^*(\Omega')b(\Omega',0)d\Omega'_{\rm cav}, \tag{25}$$

where

$$F_{\rm cav}=Z\int_{l/c}^T dtE_L^+(l,t)\int d\Omega\sqrt{\Omega}f^*(\Omega,t)\sin(\Omega l/c), \tag{26}$$

and

$$F_{\rm ext}(\Omega')=Z\int_{l/c}^T dtE_L^+(l,t)\int d\Omega\sqrt{\Omega}h^*(\Omega,\Omega',t)\times\sin(\Omega l/c). \tag{27}$$

We take T much larger than the inverse of the widths of both pulses, so that all their energy is absorbed by the detectors, which allows us to safely extend the time integration from zero to infinity in Eqs. (26) and (27). We also approximate $\sqrt{\Omega}\approx\sqrt{\omega_0}$ (this amounts to replacing the intensity by the photon flux [9,25]). We get then

$$F_{\rm cav}=\frac{2\sqrt{\gamma\Gamma}e^{i(\varphi-\zeta)}}{\sqrt{4\Delta^2+(\gamma+\Gamma)^2}} \tag{28}$$

and

$$F_{\rm ext}(\Omega)=-i\sqrt{\frac{2\pi}{\Gamma}}\frac{\Delta+\frac{i}{2}(\gamma-\Gamma)}{\Omega-\omega_L-i\gamma/2}F_{\rm cav}, \tag{29}$$

where $\Delta=\omega_L-\omega_0$ and $\zeta=\tan^{-1}(\gamma+\Gamma)/2\Delta$.

At this point it is important to note that the measured quadrature does not correspond to a single-frequency mode, but to the operator $B_\varphi+B_\varphi^\dagger$, which is a collective operator involving modes with different frequencies. The direction of the quadrature depends on the relative phase between the local oscillator and the field which left the cavity, and the Wigner function determined by the tomographic procedure is the one associated with the collective operators $B\equiv B_\zeta$ and $B^\dagger\equiv B_\zeta^\dagger$. In our model this Wigner function is given by the Fourier transform of the characteristic function,

$$C^{\rm ext}(\lambda,\lambda^*)=e^{-|\lambda|^2/2}{\rm tr}\{\rho e^{\lambda B^\dagger}e^{-\lambda^*B}\}. \tag{30}$$

If the initial density matrix operator ρ_0 is a direct product

$$\rho_0=\rho_{\rm cav,0}\otimes\rho_{\rm ext,0}, \tag{31}$$

we have

$$C^{\rm ext}(\lambda,\lambda^*)=e^{-|\lambda|^2/2}{\rm tr}\{\rho_{\rm cav,0}e^{\lambda F_{\rm cav}a_0^\dagger}e^{-\lambda^*F_{\rm cav}^*a_0}\}\times{\rm tr}\{\rho_{\rm ext,0}e^{\lambda\int F_{\rm ext}(\Omega')b^\dagger(\Omega',0)d\Omega'}\times e^{-\lambda^*\int F_{\rm ext}^*(\Omega')b(\Omega',0)d\Omega'}\}.$$

Furthermore, if the initial field is totally concentrated in the cavity, so that the initial state of the external field is the vacuum, the trace over the external modes yields the identity, leaving us with the very simple formula

$$C^{\rm ext}(\lambda,\lambda^*)=e^{(-|\lambda|^2/2)(1-|F_{\rm cav}|^2)}C^{\rm cav}(\lambda F_{\rm cav},\lambda^*F_{\rm cav}^*,0), \tag{32}$$

which relates the symmetric characteristic functions corresponding to the external field and the initial field in the cavity.

The Wigner function determined by the tomographic procedure described above is given by the Fourier transform of Eq. (32)

$$W_{\rm ext}(\eta,\eta^*)=\int e^{\eta^*\lambda-\eta\lambda^*}e^{(-|\lambda|^2/2)(1-|F_{\rm cav}|^2)}\times C^{\rm cav}(\lambda F_{\rm cav},\lambda^*F_{\rm cav}^*,0)\frac{d^2\lambda}{\pi}, \tag{33}$$

which may also be written as

$$W_{\rm ext}(\eta,\eta^*)=\int e^{[(\eta^*/F_{\rm cav})z]-[(\eta/F_{\rm cav}^*)z^*]} \times e^{s(|z|^2/2)}C^{\rm cav}(z,z^*)\frac{d^2z}{\pi|F_{\rm cav}|^2}, \quad (34)$$

where

$$s=1-\frac{1}{|F_{\rm cav}|^2}. \quad (35)$$

Cahill and Glauber [1] defined a generalized phase-space representation

$$\mathcal{W}(\alpha,\alpha^*,s)=\int e^{\alpha* z-\alpha z^*}e^{s(|z|^2/2)}C(z,z^*)\frac{d^2z}{\pi}, \quad (36)$$

labeled by a continuous ordering parameter s which can assume any value in the complex plane. In the special cases where $s=-1,0,1$, we recover the Husimi Q function, the Wigner function, and the Glauber P function, respectively. Knowledge of any of these representations over the phase space gives a complete description of the field state.

Comparing Eqs. (36) and (34) we have

$$W_{\rm ext}(\alpha F_{\rm cav}^*,\alpha^* F_{\rm cav})=\frac{1}{|F_{\rm cav}|^2}\mathcal{W}_{\rm cav}(\alpha,\alpha^*,s). \quad (37)$$

Equation (37) shows that, in principle, we may obtain complete information on the initial field inside the cavity from measurements of the quadratures $B+B^\dagger$, although what one determines in this homodyne experiment is in general the phase-space function $\mathcal{W}_{\rm cav}(\alpha,\alpha^*,s)$, and not the usual Wigner function. From Eqs. (28) and (35) we see that $|F_{\rm cav}|\leqslant 1$ and that

$$s=-\frac{(\gamma-\Gamma)^2+4\Delta^2}{4\gamma\Gamma}$$

is upper bounded by zero. Therefore the phase-space distribution measured outside the cavity is usually smoother than the Wigner function of the state originally inside the cavity. The smaller the detuning and the closer the widths of the local oscillator and the cavity mode, the closer one would be to the Wigner function of the internal field. The best result is obtained when the local oscillator field has the same frequency dependence as the cavity mode, which means $\omega_L=\omega_0$ and $\gamma=\Gamma$. In this case, $|F_{\rm cav}|$ is equal to one and the measurement of the external Wigner function reproduces exactly the Wigner function of the initial state of the field inside the cavity. Another interesting limit is obtained when $\Delta=\gamma$ and $\gamma=\Gamma$, in which case $s=-1$ and the measurement of the outside Wigner function reproduces exactly the Q function of the initial state of the field inside the cavity. In this case the complete quantum information would still be there, but it would be harder to find, since the experimental errors would play a more important role in view of the smoothness of the Q function. The best possible choice, in the sense of having maximum sensitivity to the quantum characteristics of the state, is then $|F_{\rm cav}|=1$. This result has a simple physical interpretation: the best way to determine the outgoing field by homodyne detection is to probe it with a local oscillator field that reproduces exactly its mode shape, thus matching precisely the weights given to each frequency involved in the spread of the information contained in the initial state. This behavior should be expected on the basis of the analysis of Ref. [9], where it was shown that the relevant part of the field in pulsed quantum tomography was the one mode-matched to the mode defined by the local oscillator. One should remark, however, that our aim here is quite different from the one in Ref. [9], where the authors were interested in analyzing the information which could be obtained on a running field by using pulsed tomography. Here, one is interested, along the lines of Ref. [22], in relating the information obtained on the field outside the cavity with the phase-space distribution of the internal field. That is why it was actually necessary to solve the dynamical problem in Sec. III.

It would seem from the above relations that full information on the internal field is obtained even if the local oscillator pulse does not match the outgoing field. The worst case, $s=-\infty$, corresponds to the two extreme cases of a single-mode or a very narrow local oscillator field. However, the above discussion has not taken into account the actual process of reconstructing the phase-space distribution of the field outside the cavity from homodyne measurements. In fact, the factor $|F_{\rm cav}|^2$, which measures the matching between the cavity mode and the local oscillator, plays here the role of a detection efficiency η, so that the parameter s may be expressed as $s=1-1/\eta$ [8,9]. The remarks made in Refs. [26,27] should then be applied to this case: as the parameter s gets smaller, compensation of the ''losses'' gets more difficult, and may eventually become practically impossible. Quantum details become more and more smoothed out, and the retrieval of the original quantum information becomes harder. For example, at $\eta=1/2$, corresponding to $s=-1$, we get the Husimi Q function, from which we cannot retrieve in practice the density matrix in the Fock basis due to the divergence of statistical fluctuations of the corresponding matrix elements [27].

V. EXAMPLES

Figure 2 compares the Wigner function $W_{\rm cav}$ for an initial catlike state of the field inside the cavity [15],

$$|\Psi\rangle=\frac{|\alpha\rangle+|-\alpha\rangle}{\sqrt{(2+2e^{-2|\alpha|^2})}}, \quad (38)$$

with $\alpha=3$, with the Wigner function $W_{\rm ext}$ corresponding to the external field, for zero detuning ($\Delta=0$) and $\gamma=\Gamma$ or $\gamma=0.5\Gamma$ [Figs. 2(a) and 2(b)], and also for $\Delta=\gamma=\Gamma$ [Fig. 2(c)], which corresponds to the Q function ($s=-1$). These

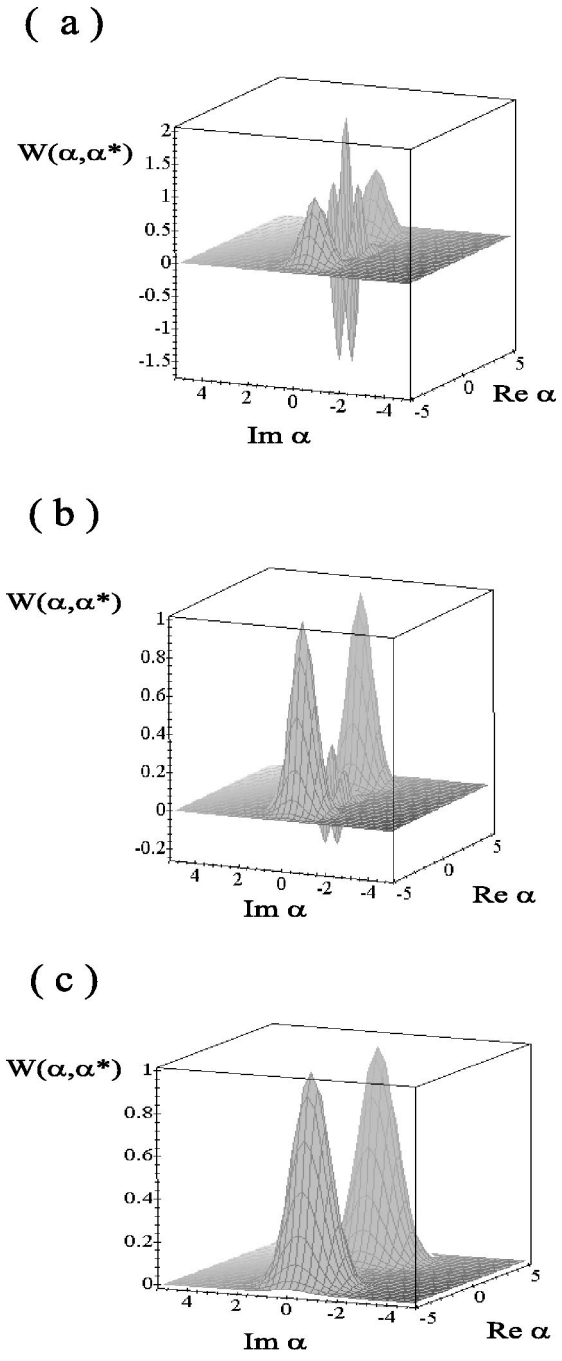

FIG. 2. Wigner function for the catlike state with $\alpha=3$ reconstructed outside the cavity when (a) $\gamma=\Gamma$ and $\Delta=0$ (in this case, the Wigner function coincides with the original one); (b) $\gamma=0.5\Gamma$ and $\Delta=0$; (c) $\gamma=\Gamma=\Delta$ (Q function).

figures clearly display the decrease in the fringe contrast, as the width or the central frequency of the homodyning pulse become different from those of the measured field.

As a second example we consider the relation between the measurement of the squeezing [28] of the outside field B and the squeezing of the initial field inside the cavity.

The squeezing in the quadrature $X_\theta^{\rm ext}=(Be^{i\theta}+B^\dagger e^{-i\theta})$ of the external field may be expressed through the parameter

$$\kappa_\theta^{\rm ext}=1-2\Delta X_\theta^{\rm ext},$$

where the variance $\Delta X_\theta=\langle X_\theta^2\rangle-\langle X_\theta\rangle^2$ may be simply related to the symmetric characteristic function $C^{\rm ext}(|\lambda|e^{-i\theta},|\lambda|e^{i\theta})$ by

$$\Delta X_\theta^{\rm ext}=\frac{d^2C^{\rm ext}(|\lambda|e^{-i\theta},|\lambda|e^{i\theta})}{d|\lambda|^2}-\left[\frac{dC^{\rm ext}(|\lambda|e^{-i\theta},|\lambda|e^{i\theta})}{d|\lambda|}\right]^2, \tag{39}$$

calculated at $|\lambda|=0$. Using Eq. (32) we obtain

$$\kappa_\theta^{\rm ext}=|F_{\rm cav}|^2\kappa_{\theta+\beta}^{\rm in}, \tag{40}$$

where $F_{\rm cav}=|F_{\rm cav}|e^{i\beta}$, and $\kappa_{\theta+\beta}^{\rm in}=1+\Delta X_{\theta+\beta}^{\rm in}$ is the squeezing in the internal field quadrature $X_{\theta+\beta}^{\rm in}=(ae^{i(\theta+\beta)}+a^\dagger e^{-i(\theta+\beta)})$. Equation (40) shows that the external squeezing $\kappa_\theta^{\rm ext}$ associated to the quadrature $X_\theta^{\rm ext}=(Be^{i\theta}+B^\dagger e^{-i\theta})$ is proportional to the squeezing $\kappa_{\theta+\beta}^{\rm in}$ in the quadrature $ae^{i(\theta+\beta)}+a^\dagger e^{-i(\theta+\beta)}$ of the initial internal field. Also, $\kappa_\theta^{\rm out}\leqslant\kappa_{\theta+\beta}^{\rm in}$. Of course, we can always make $\beta=0$ by suitably choosing the local field phase φ.

Again, the best situation, for which there is no squeezing depletion, is reached when $\Delta=0$ and $\gamma=\Gamma$. One should note that the above results coincide with those corresponding to the situation in which there is perfect mode matching but a finite detector efficiency $\eta=|F_{\rm cav}|^2$, in agreement with the discussion in [8,9].

VI. CONCLUSION

We have shown, within the framework of a simple model, that it is possible to retrieve complete information about the quantum state of a field that was originally inside a cavity, through a detailed analysis of a pulsed homodyne measurement. The Wigner function of the external field, defined in terms of collective field operators, corresponds to a generalized phase-space distribution for the original field in the cavity. When the shape of the local-oscillator pulse coincides with the shape of the emerging field, one obtains precisely the Wigner distribution for the original internal field. Otherwise, smoother distributions are obtained. For equal widths, and a detuning equal to the common width, the Q function is obtained. As examples we have discussed the measurement of a Schrödinger-cat-like state and of a squeezed field.

Previous work has derived relations between normal-ordered correlation functions of external (out) and internal (in) operators [22]. Here we have considered the relation between the quantum states of the external and internal fields. Crucial to our derivation was the operational definition of the Wigner function of the external field in terms of collective operators, which arise naturally from the analysis

of the homodyning process. This standpoint allowed us to avoid the complications associated with the continuum of modes of this field. In fact, one could have defined the external field Wigner function in terms of an infinite product of single-mode Wigner functions, which would be ill defined in the infinite-volume limit. We have shown, however, that the relevant distribution which arises from a pulsed homodyne measurement is much simpler, and remains well-defined in the continuum limit.

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